Quantum metrology: Heisenberg limit with bound entanglement

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Abstract Quantum metrology promises for a huge boost in the precision of parameters estimation. However, it requires genuine entanglement and it seems to be extremely sensitive to noise. It is still not known what type of entanglement is required for the sub-shot noise precision in quantum metrology (if the probe state should contain distillable entanglement, if it should violates local realism, etc). Here we address this problem providing a counterintuitive example of a family of bound entangled states which can be used in quantum enhanced metrology. We show that these states give advantage as big as maximally entangled states and asymptotically reach the Heisenberg limit. Moreover, entanglement of the applied states is very weak which is reflected by its so called unlockability property. Finally, we find instances where behaviour of Quantum Fisher Information reports presence of bound entanglement while a well-known class of strong correlation Bell inequality does not. The last result refers to the question if violation of local realism is required in quantum enhanced metrology. (Technical version: arXiv:1403.5867)

Introduction Estimation of a physical parameter is an important goal in many areas of science [1]. Obviously, we want to obtain the highest possible accuracy of that estimation. We can improve the accuracy repeating the experiment multiple times or, equivalently, make multipartite probe to interact with the system. In the quantum world there is another possibility of increasing the accuracy: prepare the probe in a particular quantum state i.e. in the entangled state. To be more concrete, for a classical probe that contains n particles (we can also consider it as a measurement performed n times) accuracy scales like $1/\sqrt{n}$. That is so called Shot-Noise Limit (SNL). However, if the system is in particular entangled state, then accuracy can be improved up to $1/n$. This limit, called Heisenberg Limit (HL) gives us the best what we can get that is allowed by quantum mechanics. Both of these bounds can be derived from quantum Cramer-Rao bound and Quantum Fisher information (QFI) [2–6].

Measurement resulting in improvement below shot-noise limit is a signature of quantum entanglement in the system, that is: quantum metrology technique may serve as a witness of genuine quantum correlations in the system. Indeed, how powerful it may be, we shall see below.

It is known that genuine multipartite quantum entanglement is necessary to surpass the SNL (see [8]), however not every entangled state gives the same improvement, and among entangled states there are also states that are not suitable for quantum metrology i.e. they do not surpass SNL. In particular quantum scaling is hard to obtain in case of entangled states with high noise factor (see [7]) which we have to deal with in realistic experiments where decoherence and preparation errors are present.

States on which we focus in this paper belong to a group of states with such high noise factor that makes them unusable for most of the quantum information tasks. These highly mixed states are bound entangled (BE) [9, 10]. Bound entangled states are those from which no pure entanglement can be distilled when only local operations and classical communication (LOCC) are available. The sufficient condition for entangled state to be bound entangled is its positive partial transposition [9, 11]. Among multipartite bound entangled states we distinguish unlockable and non-unlockable ones. Unlockable BE states are those, in which grouping some parties together and allowing them to perform collective quantum operations, makes distillation of pure entanglement between two other parties outside the group possible. Non-unlockable BE states are those in which we cannot obtain pure entangled state by these means. One may say that entanglement is "more bound" there.

Impossibility of pure entanglement distillation makes BE states not useful for many quantum information and communication tasks such as quantum teleportation or dense coding. In case of metrology, no instance of usefulness of BE states was known so far. In [12] the authors relate QFI and BE states and show that for certain BE states, averaged Fisher Information is higher than for separable states. However, even though the relation with averaged QFI was given, the usability of BE in case of standard formulation of quantum metrology (i.e. with known interaction between a system and a probe) remained an open question.

Our motivation here is the lack of knowledge which states are useful for the standard quantum metrology and which are not. Fitness of BE for purposes of quantum information theory (especially in the context of quantum information processing) is also not fully recognized yet. For these reasons we focus on the long-standing question "Do, among bound entangled states, there exist any examples that beat the shot-noise limit?". Intuition suggests that the high degree of noise of BE should be the reason of the negative answer. It is, in particular, especially tempting to expect such answer in classes of multiqubit states, the entanglement of which can not be unlocked.

Results We investigate a class of mixed states that are GHZ-diagonal and present the first, to our knowledge, example of bound entangled states which have advantage over product states in metrology of phase shift around z-axis. What is more, in the discussed states, the entanglement cannot be unlocked. Our family of states approaches Heisenberg limit asymptotically (an^2 scaling of the QFI with $a \geq \frac{1}{4}$). We compare QFI with multipartite Bell inequalities (as a tool of entanglement detection) and find that in some cases the sub-shot noise reports entanglement even when the well-known rich class of correlation Bell inequalities do not.

One may ask, how our states, even though highly noisy, can surpass the no-go result [7] according to which quantum scaling cannot be obtained in presence of generic local noise. The reason is the different structure of the noise in our case. In particular, our states do not have full rank, unlike in the case of generic local noise.

We consider a class of n-qbit states that are diagonal in the generalized GHZ-basis (the coefficients λ_i^+, λ_i^- will be described later):

$$\rho_{n,k} = \sum_{i=0}^{2^{n-1}-1} \left(\lambda_i^+ \left|\phi_i^+\right\rangle \left\langle\phi_i^+\right| + \lambda_i^- \left|\phi_i^-\right\rangle \left\langle\phi_i^-\right| \right). \tag{1}$$

By generalized GHZ-basis we mean:

$$\left|\phi_i^\pm\right\rangle = \frac{1}{\sqrt{2}} \left(|i\rangle \pm |\bar{i}\rangle \right), \tag{2}$$

where for n-qubit system $i \in \{0, 1, ..., 2^{n-1} - 1\}$. Here we put n-digit binary representation of i in $|i\rangle$ and its negation in $|\bar{i}\rangle$. Note that in the range of indices (i.e. $\{0, 1, ..., 2^{n-1} - 1\}$) the n-digit binary representation of i always starts with 0. For example, for 4-qubit system we have $\left|\phi_2^\pm\right\rangle = \frac{1}{\sqrt{2}} \left(|0010\rangle \pm |1101\rangle\right)$. We use the notation $\#1(i)$ for the number of ones in the binary representation of the number i. For instance $\#1(i = 7) = 3$ since the binary representation "1101" of the number 7 contains three ones.

For $\#1(i) < k$ or $\#1(i) > n - k$, the coefficient $\lambda_i^+ = \lambda$; for $\#1(i) = k$ or $\#1(i) = n - k$, the coefficient $\lambda_i^+ = \lambda_i^- = \lambda/2$; all the others λ_i^+, λ_i^- are equalt 0. Here $\lambda = 1/\sum_{i=0}^{k} \binom{n}{i}$ follows directly from the normalisation condition.

The exemplary state $\rho_{n,k}$ with $n = 4, k = 2$ is presented below

$$\rho_{4,2} = \frac{1}{11} \sum_{i \in I_1} |\phi_i^+\rangle \langle \phi_i^+| + \frac{1}{22} \sum_{i \in I_2} |\phi_i^+\rangle \langle \phi_i^+| + |\phi_i^-\rangle \langle \phi_i^-| \tag{3}$$

where $I_1 = \{0, 1, 2, 7\}$ and $I_2 = \{3, 4, 5, 6\}$.

Proposition 1. *For any n, k the state $\rho_{n,k}$ passes positive partial transpose test (PPT) with respect to local transposition on any single qubit system and, as such, it is bound entangled.*

We study usefulness of states ρ for quantum metrology in terms of Quantum Fisher Information (QFI) which quantifies the amount of information on unknown parameter θ that may be extracted by optimal measurements. In case of multipartite separable states, maximal value of QFI scales linearly with the system size (SNL). This is reflected by the separability condition for quantum Fisher information F_Q (see [12]), i.e. for any separable state ρ_{sep}, it holds:

$$F_Q(\rho_{sep}) \leq n. \tag{4}$$

On the contrary, the highest scaling i.e. quadratic one (HL) $F_Q(\rho) \approx n^2$ may be achieved only by entangled states ρ.

We obtained that for $k = 2$ and $n \geq 7$ ($k = 3$ and $n \geq 8$ respectively) $F_Q(\rho_{n,k}) > n$. Moreover we get:

Proposition 2. *Quantum Fisher Information $F_Q(\rho_{n,k})$ satisfies:*

$$F_Q(\rho_{n,k}) \geq (n - 2k)^2 \frac{k}{n+1} \tag{5}$$

for any n and $k < \frac{n}{2}$. In particular putting $k(n) = an$ ($a < \frac{1}{2}$) it follows the asymptotic behaviour

$$\lim_{n \to \infty} \frac{F_Q(\rho_{n,k(n)})}{a(1-2a)n^2} \geq 1. \tag{6}$$

We find that for the states $\rho_{n,k}$, the efficiency of Fisher information separability test does not coincide with the efficiency of some strong Bell inequality tests. We have chosen the family of all n-qubit-correlation Bell inequalities with $2^{n-1} \times 2^{n-1} \times ... \times 2^{n-2} \times ... \times 2$ settings per sides (see [14], [15]), which can be written as the simple inequality

$$\mathcal{C}^{(n)}(\rho) \leq 1 \tag{7}$$

where $\mathcal{C}^{(n)}$ is a special (optimised) function of the correlation tensor of the state. The above can be seen as a necessary condition for separability of any n-qubit state. Remember that another necessary condition for separability of n qubits is the shot-noise limit bound. We have calculated the upper bound for the factor (7) for some states from the class $\rho_{n,k}$. We obtained that for $k = 2$ and $k = 3$ and n respectively from the set $\{7, 8\}$ and $\{8, 9, 10\}$, Fisher Information criterion outperform correlation Bell inequality condition (7), i.e. it detects entanglement while correlation condition indicate hidden variable model (see the Figure 3 for the details).

The above observation suggests need of deep study of nonlocality in the context of metrology which was, to our knowledge, not pursued so far. In particular one of the questions that may be raised is possible role of metrology as a necessary condition for standard, or even weaker i.e. sequential, nonlocality.

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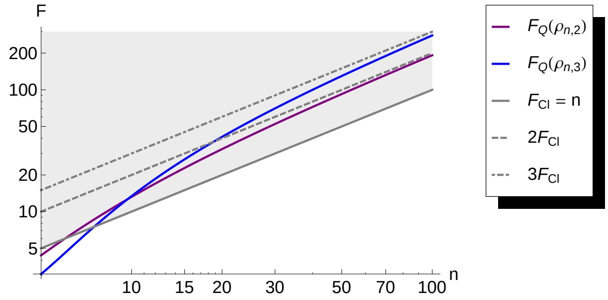

FIG. 1: Precisely calculated QFI and the limit it achieves in the infinity for two different values of k. Blue and purple lines show how the $F_Q^{n,k}$ change for $k = 3$ and $k = 2$ respectively. Dashed lines depict asymptotes. Shaded area correspond to sub-shot noise accuracy.

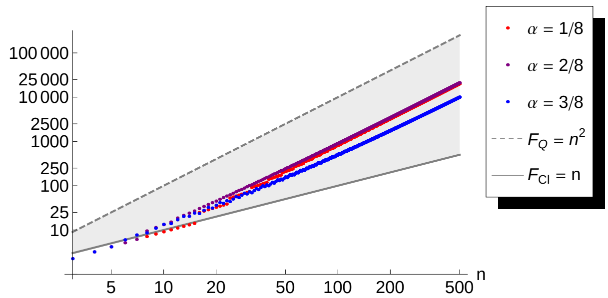

FIG. 2: Asymptotic behaviour of the quantum Fisher information for $\rho_{n,k(n)}$ states in the case when $k = an$. Dependence on n is calculated for three different values of a: $1/8$ (red), $1/4$ (purple) and $3/8$ (blue). Shaded area correspond to sub-shot noise accuracy.

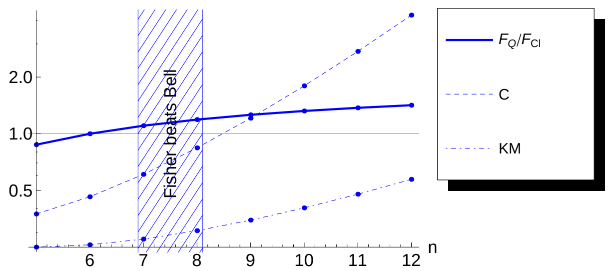

FIG. 3: Comparison of Fisher Information criterion and correlation condition in the power of entanglement detection. Here we plotted F_Q/F_{Cl} and $\mathcal{C}^{(n)}$ for $\rho_{n,2}$. Tests detect entanglement when their values exceed 1. The most interesting region is where Fisher Information criterion detect entanglement for states with hidden variable model (i.e. $F_Q/F_{Cl} > 1$ and $\mathcal{C}^{(n)} < 1$, see main text). For comparison we also plot value of Klyachko-Mermin (KM) inequality. For analysed states it performs much worse than $\mathcal{C}^{(n)}$.