Optimal State Exclusion for Symmetric Sets of States

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In the task of quantum state exclusion [1], one is given a quantum system prepared in some state ρ_x chosen from a given set $\{\rho_x\}_{x\in\mathbf{X}}$ with probability p_x. The goal of state exclusion is to find a quantum measurement $\mathsf{M} = \{M_x\}_{x\in\mathbf{X}}$ whose outcomes rule out incorrect hypotheses about the preparation of the system. More precisely, $p(x'|x) := \mathrm{Tr}[M_{x'}\rho_x]$ is interpreted as the probability of ruling out the preparation $\rho_{x'}$ given that the system is in the state ρ_x. The goal of the optimization is to find a measurement that minimizes the probability of rejecting the correct hypothesis, given by

$$p[\mathsf{M}] = \sum_{x\in\mathbf{X}} p_x \; \mathrm{Tr}[\rho_x M_x]\,. \tag{1}$$

The minimization of $p[\mathsf{M}]$ over all quantum measurements is a semidefinite program (SDP) and its dual has been investigated in Ref. [1], providing sufficient and necessary conditions for optimality. Formally, the SDP in quantum state exclusion resembles the SDP for the more familiar problem of optimal state discrimination [2–5], with the capital difference that in the former we want to minimize $p[\mathsf{M}]$, while in the latter one wants to maximize it.

The task of quantum state exclusion plays an interesting role in several areas of quantum information. Its earliest appearance is probably in the maximization of the mutual information, where measurements that achieve perfect exclusion (i.e. such that $p[\mathsf{M}] = 0$) turned out to be optimal strategies for specific sets of states. In particular, Davies [6] showed the optimality for four qubit states with tetrahedral symmetry, while Sasaki *et al* [7] proved the optimality for equally spaced qubit states on the equator of the Bloch sphere. The exclusion of quantum states played a role also in the Bayesian compatibility of quantum state assignments [8] and, more recently, in the study of limitations on hidden variable models for quantum theory [9].

In this work, we investigate the exclusion problem for sets of pure states that are generated by the action of a group representation. The interest in this scenario is motivated by the applications in Refs. [6–9], which all concern pure states with group symmetry. In this setting, we solve completely the optimization problem, providing an explicit condition for perfect exclusion and, more generally, an analytical formula for the minimum of $p[\mathsf{M}]$. Our result is then applied to a number of concrete examples: in particular, we work out the optimal exclusion strategy for sets of states generated by cyclic groups, showing that there exists a finite number N, such that perfect exclusion is possible using N independent copies.

Finally, we extend our analysis to probabilistic strategies, which take into account the possibility of an inconclusive result ?. In this setting, we prove that for symmetric sets of pure states there is always a finite probability of achieving unambiguous exclusion (i.e. of satisfying the conditions $p[\mathsf{M}] = 0$ and $p_? < 1$). Furthermore, we quantify the exact tradeoff between the probability $p[\mathsf{M}]$ and the probability of the inconclusive result.

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