Distinguishing maximally entangled states by one-way local operations and classical communication

Zhi-Chao Zhang[1], Ke-Qin Feng[2], Fei Gao[1],* and Qiao-Yan Wen[1]
[1]*State Key Laboratory of Networking and Switching Technology, Beijing University of Posts and Telecommunications, Beijing, 100876, China*
[2]*Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, China*

I. INTRODUCTION

In quantum information theory, one of the main interesting questions is whether a set of $N \leq d$ orthogonal maximally entangled states in $d \otimes d$ can be perfectly distinguished by LOCC for all $d \geq 4$.

To help answer this question, some results have been presented in Ref.[1-4]. Firstly, Bandyopadhyay *et al.* [1] gave examples of sets of d or $d-1$ maximally entangled states in $d \otimes d$ for $d = 4, 5, 6$ that cannot be perfectly distinguished by one-way LOCC. Then, Yu *et al.* [2] constructed four locally indistinguishable maximally entangled states by positive partial transpose operations (PPT operations) in $4 \otimes 4$. Furthermore, Cosentino [3] used semidefinite programming to prove there is a set of d maximally entangled states that is not perfectly distinguishable by PPT operations in $d \otimes d, d = 2^n$. Recently, we presented there exist examples of sets of $d-1$ or $d-2$ maximally entangled states in $d \otimes d$ for $d = 7, 8, 9, 10$ that are not perfectly distinguishable by one-way LOCC [4].

All the above results have only given some locally indistinguishable maximally entangled states in special quantum systems. They do not discuss the local indistinguishability in general quantum systems. Naturally, finding the general locally indistinguishable maximally entangled states in $d \otimes d$ is still meaningful and interesting.

In this paper, we focus on constructing the general one-way LOCC indistinguishable orthogonal maximally entangled states in $d \otimes d$. Fortunately, we find there are $\frac{d+5}{2}$ orthogonal maximally entangled states in $d \otimes d$ (d is odd) that cannot be distinguished by one-way LOCC. And we construct $\frac{d+4}{2}$ orthogonal maximally entangled states in $d \otimes d$ (d is even) that are not distinguishable by one-way LOCC. Furthermore, for one-way LOCC indistinguishability of the states, we present a very simple effective proof method which is based on the Fourier transform of additive group. Lastly, we present some examples for less number of maximally entangled states that cannot be perfectly distinguished by one-way LOCC. Our results show the conjectures in Ref.[4] are right.

* gaofei_bupt@hotmail.com

II. PRELIMINARIES

In $d \otimes d$, d^2 generalized Bell states can be expressed as

$$|\Psi_{nm}^{(d)}\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} e^{\frac{2\pi ijn}{d}} |j\rangle \otimes |j \oplus_d m\rangle \tag{1}$$

for $n, m = 0, 1, ..., d-1$, where $j \oplus_d m \equiv (j+m) \bmod d$. The standard maximally entangled state $|\Phi^+\rangle$ in $d \otimes d$ is $|\Psi_{00}^{(d)}\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle \otimes |j\rangle$. It is easy to verify that

$$(I \otimes U_{nm}^{(d)})|\Psi_{00}^{(d)}\rangle = |\Psi_{nm}^{(d)}\rangle \tag{2}$$

where

$$U_{nm}^{(d)} = \sum_{j=0}^{d-1} e^{\frac{2\pi ijn}{d}} |j \oplus_d m\rangle\langle j| \tag{3}$$

$U_{nm}^{(d)}$ is $d \times d$ unitary matrices for $n, m = 0, 1, ..., d-1$.

Definition 1. Let $S_k^{(d)} = \{|\Psi_i^{(d)}\rangle = (I \otimes U_i)|\Psi_{00}^{(d)}\rangle, i = 0, \ldots, N-1\}$ be a set of N number of one-way LOCC indistinguishable maximally entangled states in $d \otimes d$, we define $f(d) = \min\{|S_k^{(d)}|, k = 1, 2, \ldots\}$.

Lemma 1 [4]. In $d \otimes d$, $N \leq d$ number of pairwise orthogonal maximally entangled states $|\Psi_{n_i m_i}^{(d)}\rangle$ (for $i = 1, 2, ..., N$), taken from the set given in Eq. (1), can be perfectly distinguished by one-way LOCC, if and only if there exists at least one state $|\alpha^{(d)}\rangle$ for which the states $U_{n_1 m_1}^{(d)}|\alpha^{(d)}\rangle, U_{n_2 m_2}^{(d)}|\alpha^{(d)}\rangle, ..., U_{n_N m_N}^{(d)}|\alpha^{(d)}\rangle$ are pairwise orthogonal, where $U_{n_i m_i}^{(d)}$'s are given by Eq. (3).

Lemma 2. Letting $F : \mathcal{Z}_d \to \mathbb{C}$ is a complex valued function of additive group $\mathcal{Z}_d = \{0, 1, \ldots, d-1\}$, the Fourier transform of F is $M : \mathcal{Z}_d \to \mathbb{C}$, where $M(n) = \sum_{j=0}^{d-1} \omega_d^{jn} F(j)$, $0 \leq n \leq d-1$, $\omega_d = e^{\frac{2\pi i}{d}}$. Defining the subset of $\mathcal{Z}_d$ is $A = \{n \in \mathcal{Z}_d : M(n) = 0\}$, then we know $F(j) = \frac{1}{d} \sum_{n \in \mathcal{Z}_d \setminus A} \omega_d^{-jn} M(n)$, $0 \leq j \leq d-1$.

III. ONE-WAY LOCC INDISTINGUISHABLE OF ORTHOGONAL MAXIMALLY ENTANGLED STATES

In this section, we construct the orthogonal maximally entangled states in the quantum systems of $d \otimes d$ and prove these states are indistinguishable by one-way LOCC. Then, we present an upper bound for the number of one-way LOCC indistinguishable orthogonal maximally entangled states as follows.

Theorem 1. In $d\otimes d, d\geq 2$, (1) when d is odd, $f(d)\leq \frac{d+5}{2}$; (2) when d is even, $f(d)\leq \frac{d+4}{2}$.

Proof. (1). When d is odd, letting $N=\frac{d+5}{2}$, we consider the set $S_1^{(d)}=\{|\Psi_{n_im_i}^{(d)}\rangle=(I\otimes U_{n_im_i})|\Psi_{00}^{(d)}\rangle, i=0,\ldots,N-1\}$, where $(n_i,m_i)=(i,0)$ for $i=0,\ldots,N-3$, $(n_{N-2},m_{N-2})=(0,1)$, $(n_{N-1},m_{N-1})=(0,d-1)$. We can get that $S_1^{(d)}$ is one-way LOCC indistinguishable.

(2). When d is even, letting $N=\frac{d+4}{2}$, we construct the set $S_2^{(d)}=\{|\Psi_{n_im_i}^{(d)}\rangle=(I\otimes U_{n_im_i})|\Psi_{00}^{(d)}\rangle, i=0,\ldots,N-1\}$, where $(n_i,m_i)=(i,0)$ for $i=0,\ldots,N-2$, $(n_{N-1},m_{N-1})=(0,\frac{d}{2})$. In the same way, we can prove $S_2^{(d)}$ cannot be distinguished by one-way LOCC. ■

IV. DISCUSSION

Our results give a general unified upper bound for the number of maximally entangled states that are not perfectly distinguishable by one-way LOCC. However, the upper bound may be not a supremum for some high dimensional systems.

According to Theorem 1, we get $f(5)\leq\frac{5+5}{2}=5$. In fact, there are 4 one-way LOCC indistinguishable maximally entangled states in $5\otimes 5$ [1]. Thus, $f(5)\leq 4$. In addition, there are also other less number of maximally entangled states that are not perfectly distinguishable by one-way LOCC than that referred in Theorem 1. For example, $S_3^{(10)}=\{|\Psi_{00}^{(10)}\rangle,|\Psi_{10}^{(10)}\rangle,|\Psi_{30}^{(10)}\rangle,|\Psi_{60}^{(10)}\rangle,|\Psi_{05}^{(10)}\rangle,|\Psi_{55}^{(10)}\rangle\}$ is a set of one-way LOCC indistinguishable maximally entangled states.

In the same way, we get $S_4^{(12)}=\{|\Psi_{00}^{(12)}\rangle, |\Psi_{10}^{(12)}\rangle,|\Psi_{30}^{(12)}\rangle,|\Psi_{40}^{(12)}\rangle,|\Psi_{80}^{(12)}\rangle,|\Psi_{06}^{(12)}\rangle,|\Psi_{66}^{(12)}\rangle\}$ and $S_5^{(14)}=\{|\Psi_{00}^{(14)}\rangle,|\Psi_{10}^{(14)}\rangle,|\Psi_{30}^{(14)}\rangle,|\Psi_{50}^{(14)}\rangle,|\Psi_{60}^{(14)}\rangle,|\Psi_{07}^{(14)}\rangle, |\Psi_{77}^{(14)}\rangle\}$ are also two sets of one-way LOCC indistinguishable maximally entangled states. Therefore, $f(12)\leq 7<\frac{12+4}{2}=8$, $f(14)\leq 7<\frac{14+4}{2}=9$. Then, we understand there may be smaller upper bound for the number of one-way LOCC indistinguishable orthogonal maximally entangled states.

So far, we have not known whether the constructed indistinguishable orthogonal maximally entangled states by one-way LOCC in $d\otimes d$ can be distinguished by LOCC. It is left as an interesting open question whether there is a supremum for the number of indistinguishable maximally entangled states by one-way LOCC.

[1] S. Bandyopadhyay, S. Ghosh, G. Kar, New J. Phys. **13**, 123013 (2011).
[2] N. K. Yu, R. Y. Duan and M. S. Ying, Phys. Rev. Lett. **109**, 020506 (2012).
[3] A. Cosentino, Phys. Rev. A **87**, 012321 (2013).
[4] Z.-C Zhang, Q.-Y Wen, F. Gao, G.-J Tian and T.-Q Cao, Quantum Inf. Process. **13**, 795 (2014).