Universal transversal gates with color codes — a simplified approach

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We present a simplified yet rigorous explanation of the ideas from Bombín's paper *Gauge Color Codes* [1]. Our discussion is self-contained, and assumes only basic concepts from quantum error correction. We provide an explicit construction of a family of color codes in arbitrary dimensions and describe some of their crucial properties. Within this framework, we explicitly show how to transversally implement the generalized phase gate $\overline{R}_m = \mathrm{diag}(1, e^{2i\pi/2^m})$, which deviates from the method in Ref. [1], allowing an arguably simpler verification. We describe how to implement the Hadamard gate $\overline{H}$ fault-tolerantly using code switching technique. In three dimensions, this yields, together with the transversal $\overline{CNOT}$, a fault-tolerant universal gate set $\{\overline{H}, \overline{CNOT}, \overline{R}_3\}$ without state-distillation.

I. INTRODUCTION

A. Motivation

To build a fully functioning quantum computer, it is necessary to encode quantum information to protect it from noise. In physical systems one expects noise to act locally. Therefore, *topological codes* [2–5], which naturally protect against local errors, represent our best hope for storing quantum information. However, a quantum computer must also be capable of processing this information. This motivates the search for topological codes in which one can implement a sufficiently diverse set of gates which (i) can operate in the presence of typical noise without corrupting the stored information, and (ii) can generate any computation on the encoded information.

There are known examples of topological codes in which one can find such a set of gates, however, most require an enormous amount of overhead. For instance, using the the toric code with magic state distillation [6] requires many additional qubits [7], and computing by braiding non-abelian anyons [2] in systems with rich topological order requires additional time to ensure the anyons are moved sufficiently slowly. Such additional overhead, although polynomial, is undesirable since it could make a quantum computer orders of magnitude less efficient, both in terms of its size and its run time. Cutting-edge technology operates at the level of tens of qubits, so for the foreseeable future, large polynomial overhead will be an insurmountable problem for quantum hardware [7–9].

Here we focus on a new proposal by Bombín [1] in which there appears to be no such additional overhead. More precisely, we give a clear argument that in a three-dimensional color code with macroscopic code distance, there is a fault-tolerant, universal gate set without the need for large numbers of ancilla qubits or run time. However, this improvement comes at a price: a lattice of at least three dimensions is required, limiting this construction's practicality for reasons of architecture.

B. No-go theorems for transversal universal gate set

Fault-tolerant gates do not spread typical errors into uncorrectable ones [10]. The simplest type of fault-tolerant gates are *transversal*, meaning they are implemented by applying physical unitaries supported on single physical qubits. Unfortunately, due to a no-go theorem by Eastin and Knill [11], having a transversal universal gate set is not possible. More precisely, for any code capable of protecting against single-qubit errors, the set of transversal encoded gates preserving the code space forms a finite group, and thus cannot be universal. In the context of *local topological stabilizer codes* [12, 13] in D dimensions, Bravyi and König [14] showed that this group must be contained in $\mathcal{P}_D$ — the D^{th} level of the *Clifford hierarchy* [15]. These results have been extended in Refs. [16] and [17]. To date, the only family of local topological stabilizer codes known to saturate the Bravyi-König classification is the family of *color codes*, introduced by Bombín and Martin-Delgado [4].

C. Circumventing the no-go theorems

Despite the no-go results [11, 14, 16], there are a few ways of achieving a universal gate set for topological codes. The most well-known method involves magic-state distillation [6], but dramatically increases the overhead required for computation [7]. Another approach, suggested by Paetznick and Reichardt [18] and generalized in Ref. [19], uses the technique of *gauge-fixing* to fault-tolerantly switch between different codes, allowing one to take advantage of the properties of each. Recently, Bombín suggested that with two different color codes in three dimensions and using code switching, one can achieve a universal gate set [1]. In our article, we present a simplified yet rigorous explanation of Bombín's ideas.

II. SUMMARY OF MAIN RESULTS

We give an explicit construction for lattices with special properties in D dimension, and prove that for every such lattice, with qubits placed at vertices

1. One can always define a family of distinct subsystem color codes $CC_D(x,z)$, enumerated by pairs of integers $x,z \leq D$ satisfying $x+z \geq D+2$.
2. The logical $\overline{CNOT}$ is transversal in any color code.
3. The logical Hadamard $\overline{H}$ is transversal in a self-dual color code, $CC_D(D,D)$.
4. The logical phase gate $\overline{R}_m = \mathrm{diag}(1, e^{2i\pi/2^m})$ is transversal in $CC_D(x,z)$, where $m \leq \lfloor \frac{D}{D-x+1} \rfloor$. In particular, for $CC_D(D,2)$ one can implement $\overline{R}_D$ transversally.
5. Using code switching and gauge fixing, one can switch fault-tolerantly between $CC_D(x,z)$ and $CC_D(x',z')$ defined on the same lattice, provided $x \leq x'$ and $z \leq z'$.

A consequence of these results is that a universal gate set $\{\overline{H}, \overline{CNOT}, \overline{R}_3\}$ can be implemented fault-tolerantly with a color code in three dimensions, $CC_3(3,2)$. In this code, $\overline{CNOT}$ and $\overline{R}_3$ can be implemented fault-tolerantly, and one can switch to and from $CC_3(3,3)$ in order to apply $\overline{H}$.

Although these conclusions are also reached in Bombín's paper *Gauge Color Codes* [1], we provide alternative, rigorous and substantially simplified proofs. In particular, we provide an explicit transversal implementation of $\overline{R}_D$ gate in $CC_D(D,2)$.

We also explain how a subfamily of quantum Reed-Muller codes [19–21] can be viewed as a special case of color codes.

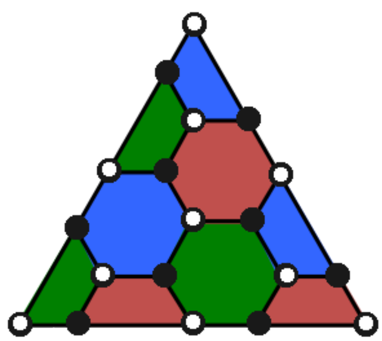

FIG. 1. Color code (CSS stabilizer code) in two dimensions. We place qubits at vertices. For each plaquette in the lattice, there is a X- and Z-type stabilizer generator with support on the vertices of that plaquette. Vertices and edges create a bipartite graph.

III. METHODS

In this section we will give a flavor of how to construct a color code and how the main results can be deduced.

A. Construction and properties of color codes

Color codes are topological CSS subsystem codes [22–25] specified by their *gauge group* $\mathcal{G}$. The gauge group is a subgroup of the Pauli group on the physical qubits $\mathcal{Q}$, and the *stabilizer group* $\mathcal{S} \subseteq \mathcal{G}$ is defined as the center of $\mathcal{G}$.

Color codes [4, 26, 27] can be defined on D-dimensional lattices which satisfy certain requirements, for instance their dual lattice must be a homogenous simplicial D-complex [28]. A D-dimensional color code is uniquely specified by its lattice and gauge group generators. We can enumerate color codes in D dimensions using pairs of integers $x, z \leq D$, satisfying $x + z \geq D + 2$. By $CC_D(x, z)$ we denote a D-dimensional color code with X- and Z-type gauge generators supported on x- and z-cells. As an example, we present a two-dimensional stabilizer color code, $CC_2(2, 2)$. (see Fig. 1).

The lattice requirements for the two-dimensional color code are as follows

- colorability — faces can be colored with 3 colors: red, green and blue, such that every two faces sharing an edge have different colors,

- valence — every vertex is 3-valent, meaning it belongs to exactly 3 edges.

The lattices satisfying the above two conditions have the following properties

Property 1. *Intersection of two faces contain even number of vertices.*

Property 2. *If the lattice has trivial first homology group, then the graph made up of all vertices and edges is bipartite.*

From Proposition 1, we see that the stabilizer generators (supported on faces) all commute as required. We rely heavily on Proposition 2 in the construction of $\overline{R}_2$, which is implemented by applying R_2 to qubits in one partite set, and R_2^{-1} to qubits in the other (see Fig. 1).

B. Switching between color codes

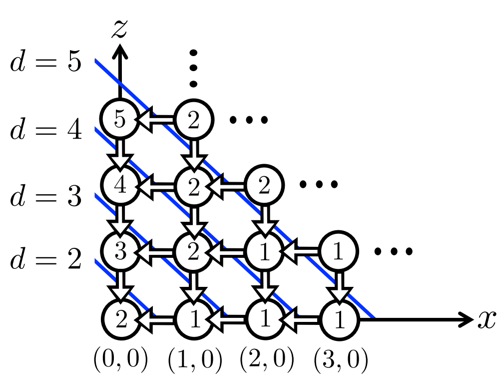

FIG. 2. Family of color codes. The point (x, z) corresponds to a color code $CC_D(D - x, D - z)$. The only color codes realizable in D dimensions are below the D^{th} diagonal line. Arrows indicate partial order. The encircled numbers indicate maximum $\overline{R}_m$ implementable for a corresponding stabilizer color code.

To evade the no-go theorem of Eastin and Knill [11], we use the idea of code switching between color codes. This will allow us to implement the transversal gates of multiple codes, and thus achieve a transversal universal gate set.

We define a *partial order* for color codes on the same lattice, such that $C \prec C'$ holds if each codeword of code C is also a codeword of C':

$$CC_D(x, z) \prec CC_D(x', z') \iff x \leq x' \wedge z \leq z'.$$

One can switch between two codes if $CC_D(x, z) \prec CC_D(x', z')$, namely

- $CC_D(x, z) \mapsto CC_D(x', z')$: one does nothing (codewords of $CC_D(x, z)$ are codewords of $CC_D(x', z')$),

- $CC_D(x', z') \mapsto CC_D(x, z)$: one can view the codewords of $CC_D(x, z)$ as those for $CC_D(x', z')$ with the additional gauge qubits present in $CC_D(x, z)$ set to a particular state. To switch, one fixes the state of the additional gauge qubits to the appropriate state [18, 19].

C. Transversal gates in color codes

Since color codes are CSS codes, the logical gate $\overline{CNOT}$ can be implemented transversally for any color code, whereas the dressed logical Hadamard gate $\overline{H}$ can be implemented when the X- and Z-type gauge generators coincide, $CC_D(d, d)$. The maximum gate $\overline{R}_m$ implementable transversally with a color code $CC_D(x, z)$ is defined by $m = \left\lfloor \frac{D}{D-x+1} \right\rfloor$, as illustrated in Fig. 2 for stabilizer color codes. Implementation of the logical $\overline{R}_m$ involves application of the gate R_m^k to qubits in one partite set, and R_m^{-k} to qubits in the other (see Fig. 1).

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