Classification of transversal gates in qubit stabilizer codes (QIP extended abstract, arXiv:1409.8320)

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I. SUMMARY

Transversal operations in quantum error correction, that is logical gates that are executed by applying a set of gates in parallel, are the most straightforward form of fault-tolerant quantum logic. Naturally, characterizing the set of logical gates that can be implemented transversally for a class of codes is of great importance to fault-tolerant quantum architectures. In this work, we classify the set of logical gates that be realized by applying individual diagonal gates transversally. Namely, we show that for qubit stabilizer subspace codes, the set of logical gates that can be implemented by an operation of the form $Z(\theta)^{\otimes n}$ are restricted to have entries of the form $e^{i\pi c/2^k}$ along their diagonal, where c is an integer. We show that these results imply that logical gates implemented in this manner must belong to the Clifford hierarchy for all stabilizer codes and moreover the single and multi-qubit gates are restricted to be elements of the same level of the Clifford hierarchy.

II. INTRODUCTION AND BACKGROUND

Quantum error correction and fault-tolerance are vital to the implementation of a coherent quantum computing device. The former addresses the ability to detect and correct given sets of error defined by the characteristics of the quantum error correcting code [1–4] while the latter is concerned with limiting the propagation of physical errors that occur in order to remain correctable [5–9]. Logical gates are applied for a given code by applying a sequence of gates on the qubits encoding the quantum information in the quantum error correcting code. The most natural form of fault-tolerant quantum gate is a transversal gate, that is a gate that is applied in parallel to individual qubits in a logically encoded codeblock, or individual pairs of qubits in the case of a logical gate between codeblocks. As such, developing quantum error correcting codes with transversal logical gates is of great interest to building a fault-tolerant quantum architecture. Unfortunately, it has been shown that there exists no quantum error correcting code that exhibits a set of universal logical gates that can be implemented transversally [10, 11]. In order to sidestep this restriction, techniques such as magic state distillation [12], gauge fixing [13, 14], fault-tolerant code concatenation [15], and code conversion [16] have been developed to implement a universal set of fault-tolerant quantum gates, each of which exploit the transversal nature of sets of logical gates in specifically chosen codes. Therefore, having an understanding of the set of logical gates that can be implemented transversally would profoundly impact multiple avenues for fault-tolerant quantum computing architectures. Finally, there has been a recent push in the quantum gate synthesis community to consider new sets of universal single-qubit gates for the purposes of gate decomposition of an arbitrary single-qubit unitary. Namely, advantages over the traditional $\{H, T\}$ basis, where H is the Hadamard gate and T is the $\pi/8$ gate ($T = \mathrm{diag}(1, e^{i\pi/4})$), have been found in the form of the $\{H, V\}$ basis, where $V = \mathrm{diag}(1 - 2i, 1 + 2i)/\sqrt{5}$, where the overall number of non-Clifford gates is reduced for decompositions in the latter basis [17]. Therefore, if it were possible to find codes that could implement the V gate transversally, this could point to more efficient fault-tolerant implementations of such a gate decomposition, as outlined by the techniques above.

Zeng *et al.* showed that any single-qubit logical gate that is implemented transversally for a qubit stabilizer code must be equivalent up to local Clifford gate operations to a transversal application of diagonal gates [11]. As stated below, this work fully classifies this set of gates and as such classifies all transversal operations for single-qubit logical

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operations up to local Clifford equivalences. Finally, it should be noted that in a parallel works transversal gates were established to belong to the Clifford hierarchy [18, 19]. Our result differs from their result primarily by the following: we establish the allowable underlying physical gates that can be used in the application of a nontrivial transversal logical gate and additionally characterize the set of diagonal two-qubit transversal gates that can be implemented between two codeblocks encoded in the same code. Importantly, we show that the diagonal two-qubit transversal gates must belong to the the same level of the Clifford hierarchy as those realizable for a single codeblock for a given code.

III. MAIN RESULT

Before arriving at the main result of the paper, we first introduce some definitions. Let $\mathcal{S} = \langle G_i \rangle_{i=1}^m$ be a stabilizer group generated by a set of m Pauli operators $\langle G_i \rangle_{i=1}^m$ on n qubits and let $\mathcal{C}_\mathcal{S} = \{|\psi\rangle \mid g|\psi\rangle = |\psi\rangle \ \forall \ g \in \mathcal{S}\}$ be the stabilizer code corresponding to the "+1" eigenspace of all of the stabilizers in $\mathcal{S}$. Such a stabilizer code will have $(n-m)$ logical qubits, and a corresponding set of logical Pauli $X_{L,i}$ and $Z_{L,i}$ operators, which are composed of a tensor product of Pauli operators [20, 21]. A CSS stabilizer code is a stabilizer code whose generators can be expressed in two distinct sets, those containing only Pauli X operators and those containing only Pauli Z operators [22, 23].

We define a strongly transversal gate as a gate of the form $Z(\theta)^{\otimes n}$, that is an identical Z basis rotation for each of the individual qubits of the code, where $Z(\theta) = \mathrm{diag}(1, e^{i\pi\theta})$. For many quantum codes, a logical gate can be realized by applying such strongly transversal gates, such as the logical phase gate for the 7-qubit Steane code or the logical T gate for the 15-qubit Reed-Muller code, both of which are CSS codes. Our first main result classifies the set of strongly transversal gates that can be implemented for nontrivial CSS codes, where nontrivial is defined to be a logical codespace of at least a single logical qubit and a code distance of at least 2 (can detect an arbitrary single qubit error).

Theorem 1. *A nontrivial CSS code can only have strongly transversal $Z(\theta)$ rotations which are of the form $Z(a/2^k)$.*

It is worth pointing out that the Reed-Muller family of quantum error correcting codes exhibit logical gates of the form $Z(a/2^k)^{\otimes n}$. The theorem is proven by considering the expansion of the logical states as a sum of states in the computational basis and imposing modular arithmetic restrictions on the weight of the elements of the stabilizer group (in terms of the binary representation of the X components of the Paulis) in order for such a gate to be a logical gate. These restrictions only allow very selective rotations in order for all of the conditions on the weights of the stabilizer group elements to be satisfied.

In addition, in this work we show a further result on the classification of diagonal rotations in the Clifford hierarchy. The Clifford hierarchy is defined recursively, beginning with the first level $\mathcal{C}_n^{(1)} = \mathcal{P}_n$ being the n-qubit Pauli operators, and the elements of subsequent levels being defined as follows: $\mathcal{C}_n^k = \{U \in U(2^n) \mid UPU^\dagger \in \mathcal{C}_n^{(k-1)} \ \forall P \in \mathcal{P}_n\}$, where $U(2^n)$ is the set of unitaries on n qubits.

Proposition 2. *Let $A = Z(\theta)$ be a diagonal single-qubit operator. If $\theta = c/2^k$, for any integer $k \geq 0$ where θ is in its most reduced form, then $A \in \mathcal{C}_1^{(k+1)}$. Otherwise, A is not in the Clifford hierarchy, that is $A \notin C_1^{(k)}$ for all k.*

Therefore, the logical gates that can be implemented in a strongly transversal manner are restricted to be composed of individual physical gates belonging to the Clifford hierarchy and strengthens the strong relationship between the Clifford hierarchy and fault-tolerance. Moreover, since the induced phase on the logical state must be a multiple of the phases of the individual qubits, the resulting logical gate must also belong to the Clifford hierarchy.

In a similar manner to the proof of the gate restrictions for CSS codes, we then extend our proof to incorporate all stabilizer codes, making extensive use of the binary representation of n-qubit Pauli operators as $2n$-bit strings. Our main result of the paper is:

Proposition 3. *A nontrivial qubit stabilizer codes can only have strongly transversal $Z(\theta)$ rotations which are of the form $Z(a/2^k)$.*

While strongly transversal gates are commonly used for the purposed of applying a logical gate, one could envision implementing a logical gate by applying individual rotations of different strength, that is rotations of the form $Z_L(\theta) = Z(\theta_1) \otimes Z(\theta_2) \otimes \ldots \otimes Z(\theta_n)$. We show that such logical gates are restricted by the same set of conditions in two steps. First, in order for such a gate to be a nontrivial (non-identity) gate the rotations must be rational (shown in the Appendix of the main document). Then, having found a common denominator, the above gate is realized by applying the same basic rotation to all physical qubits, where the rotation is applied a different number of times to each individual qubit. Finally we show by finding an appropriate distance 2 code, we can recover the proof for the case of the strongly transversal gates and that the individual rotations on the physical qubits must be of the form $Z(a/2^k)$.

Finally, we conclude by considering multi-qubit transversal gates, and the imposed restrictions on the set of two-qubit diagonal gates for two blocks of the same quantum error correcting code. In addition to the set of restrictions on the weight of the binary string expansion of the stabilizer group elements in the case of the single-qubit logical gate, there is a further restriction on the difference of the phase angles composing the multi-qubit diagonal gate. This extra restriction imposes that if in the single qubit case the code was chosen to have the ability to implement transversal diagonal gates with entries of the form $e^{i\pi/2^k}$, then in the multi-qubit case one of these phases would have to be of the form $e^{i\pi/2^{k-1}}$. The implication of this restriction is that the two-qubit logical gate that can be implemented in such a manner must reside at the same level of the Clifford as the single qubit logical gate for the given code. A compressed statement of the Theorem is as follows:

Theorem 4. *Given a CSS code that can implement the logical gate $Z_L(1/2^k)$ by applying a transversal $Z(1/2^k)^{\otimes n}$ on the underlying physical qubits yet cannot implement the gate $Z_L(1/2^{k+1})$ due to code constraints. Then, the set of two-qubit diagonal gates $U = \sum_j e^{i\pi\theta_j}$ that can implement a logical two-qubit operation by applying such gates transversally ($U^{\otimes n}$) will be restricted to be contained at the same level in the Clifford hierarchy as the single-qubit logical gate, that is $Z_L(1/2^k) \in \mathcal{C}_1^{(k+1)}$ and $U^{\otimes n} \in \mathcal{C}_2^{(k+1)}$.*

This result can be further generalized as in the single-qubit case to general stabilizer codes and can account for differing rotations on the pair of qubits. The result implies that given a stabilizer code, there will be a level in the Clifford hierarchy imposed by the code that will contain all possible nontrivial diagonal transversal logical gates, regardless of whether they are single-qubit or multi-qubit rotations.

As outlined in the Introduction & Background, the result of these restrictions are far reaching. They imply that for universal fault-tolerant techniques such as gauge fixing, code concatenation, or conversion between two sets of codes with differing transversal gate sets, the only non-Clifford gates that can be implemented transversally already all belong to a known family of codes, the Reed-Muller code family. While such codes may not be optimal in terms of the number of overall qubits and the search for other codes with similar properties could be fruitful, considering a completely different set of operations for universal logic would not be possible.

Further, for the purposes of magic state distillation, which relies on transversal logic in order to arrive at a decoding sequence for the purposes of distillation for certain (but not all) schemes, this result implies that the set of magic states that can be distilled via traditional distillation methods are limited to be of a particular form imposed by these conditions.

Finally, in the case of unitary gate decomposition for circuit synthesis, more efficient techniques based of a decomposition in terms of the $\{H, V\}$ basis may suffer from having to use approximative techniques in order to implement the V gates fault-tolerantly. Namely, since the V gate does not belong to the Clifford hierarchy it will not be implementable in a transversal way, and moreover developing a state distillation scheme to implement V exactly would not be possible via the traditional distillation methods. Therefore, to implement it logically, an appriximative state distillation technique would have to be used [24].

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