Super-activation of quantum reference frames

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Abstract

Quantum particles with spin are the most elementary gyroscopes existing in nature. But what happens when two such gyroscopes are used by two distant observers to find out their relative orientation in space? Here we show that a pair of gyroscopes in an EPR entangled state gives little clue about the relative orientation, but when two or more identical pairs are available, suddenly the error drops with the size of the system, at a rate that beats the best classical scaling already for small number of copies. This activation phenomenon indicates the presence of a latent resource hidden into EPR correlations, which can be unlocked and turned into advantage when multiple copies are available.

Information about reference frames is carried by physical systems, such as clocks and gyroscopes for time and spatial orientation. The field of quantum reference frames [1] aims at investigating how the rules of quantum mechanics affect the ability of systems to carry reference frame information [2, 3, 4, 5, 6, 7, 8], and how they can be harnessed to achieve novel information-theoretic protocols [9, 10, 11, 12, 13, 14, 15]. Here we point out a new, rather counterintuitive phenomenon, which arises when entangled states are used to encode three orthogonal directions in space. In a nutshell: a quantum gyroscope can indicate three spatial directions with a precision that does not scale with the size of the system, but when more identical copies are used, one can achieve vanishing error, with a scaling that becomes highly non-classical already for small number of copies. This phenomenon is best described in a bipartite scenario: Suppose that initially Alice and Bob are at a ground station, with their axes aligned ($\mathbf{n}_i^A \equiv \mathbf{n}_i^B \equiv \mathbf{n}_i$ for $i = x, y, z$). Then, they travel to two distant satellite stations and during the journey their local reference frames undergo two unknown rotations g_A and g_B, respectively. At this point, their task is to realign the axes, performing two rotations h_A and h_B such that $h_A g_A = h_B g_B$. We assume that only classical communication is possible between the two stations, e. g. via radio signals. Hence, Alice and Bob can only rely on the correlations between their gyroscopes: for example, if the gyroscopes are aligned with the axes at the ground station, Alice and Bob can try to estimate their orientation and to align themselves with it. In general, a perfect alignment will not be possible, due to the finite size of the gyroscopes. One way to quantify the error is to consider the square distance between Alice's and Bob's axes after the realignment, averaged over the three axes [2, 4, 5, 6, 9]

$$e(h_A, h_B, g_A, g_B) = \frac{1}{3} \sum_{i=x,y,z} \|h_A g_A \mathbf{n}_i - h_B g_B \mathbf{n}_i\|^2 . \quad (1)$$

In general, an alignment protocol will consist of local operations (LO), performed at the satellite stations, coordinated by classical communication (CC) between the two stations. Since eventually the protocol outputs a classical description of the two rotations h_A and h_B, one can describe it by a LOCC measurement. The goal of the measurement is to minimize the expected value of the error, in the worst case scenario over all possible rotations g_A and g_B.

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We now exhibit a family of quantum states where the alignment error remains constant with the size of the system. Alice's (Bob's) quantum gyroscope is a composite system, consisting of a spin-j particle, denoted by A_1 (B_1), and of a $(2j+1)$-dimensional, rotationally invariant degree of freedom, denoted by A_2 (B_2). The latter can be realized e. g. by the charge or current states of a solid state quantum device, or can be simulated by a virtual subsystem of a set of spin-$1/2$ particles [9]. For the joint state of Alice's and Bob's gyroscopes, we choose

$$\rho_{AB} = |S_j\rangle\langle S_j|_{A_1B_1} \otimes |\Phi_j^+\rangle\langle\Phi_j^+|_{A_2B_2}\,, \tag{2}$$

where $|S_j\rangle$ is the spin-j singlet [16] and $|\Phi_j^+\rangle$ is the standard maximally entangled state in dimension $2j+1$. Strictly speaking, only the systems A_1 and B_1 act as "gyroscopes" (carriers of directional information), while the role of systems A_2 and B_2 is to allow Alice and Bob to simulate a global measurement on A_1B_1.

In order to evaluate the precision of alignment, we convert the minimization of the error into a problem of quantum estimation in the presence of shared reference frames. Using this technique, we show that *i)* the best alignment protocol using the state ρ_{AB} has alignment error $e = 4/3$, independently of j and *ii)* using the state $\rho_{AB} \otimes \rho_{AB}$ the error can be reduced to the Heisenberg limit (HL) $1/j^2$ with probability p larger than 43.9%, and to the standard quantum limit (SQL) $1/j$ with probability $1-p$. For large j, this means that the state (2) has a potential to give highly precise information about three directions in space, but this potential cannot be detected for a single copy: it shows up only when $n \geq 2$ copies are available.

The activation phenomenon highlighted here is not an artifact of the specific error function used in our calculation. For example it can be observed also for the variance of the three Euler angles: for a single copy, if the sum of the three variances vanished with j, then also the average of the error (1) would have to vanish, in contradiction with result *i*). On the other hand, the fact that for two copies the error (1) vanishes implies that also the variances must vanish. The same arguments can be used for every function of the Euler angles that is sufficiently regular (basically, every function that admits a Taylor expansion up to the third order).

Going from one to two copies enhanced the precision from constant scaling to the SQL and, with good chance of success, to the HL. The probability of reaching the HL can be further amplified by repetition of the protocol, which allows one to attain HL precision with probability $p_n > 1 - (0.551)^n$ using $2n$ copies. However, one can do even better: taking advantage of joint measurements, the HL can be achieved with probability 1 using only four copies. For three copies, the HL cannot be achieved deterministically, but, interestingly enough, one can still obtain the quasi-Heisenberg scaling $e = \ln(6j)/(8j^2) + O(1/j^2)$.

We now discuss briefly some consequences and applications of our results:

Secret sharing of a reference frame. Our result suggests a way to distribute the ability to align Cartesian frames over different parties, in a way that is akin to secret sharing [18, 19]. Imagine that, in order to accomplish a desired task, the two satellite stations A and B must have their reference frames aligned with high precision. At the two stations there two groups of parties, with each pair of parties (A_i, B_i) possessing two systems in the state ρ_{AB}. Now, our result guarantees that a single pair alone cannot achieve the desired task: at least two parties have to cooperate in order to reduce the error down to zero. Moreover, if the task requires the error to be of order $1/j^2$ (instead of $1/j$ or $\log j/j^2$), then at least four parties at each station have to cooperate. Regarding the problem of secret sharing of reference frames, our protocol improves over the state of the art [14], allowing one to achieve the Heisenberg limit. Indeed, the precision of the alignment in Ref. [14] was bound to the SQL by the fact that the protocol used n spin-$1/2$ particles in a separable state.

Quantum metrology with spin-j singlets. Although we presented our results in a bipartite communication scenario, it is immediate to translate them into the conventional single-party scenario of quantum metrology. In this translation, the problem is to estimate an unknown rotation g from n copies of the rotated spin-j singlet $|S_{j,g}\rangle$, a situation that arises e.g. in the measurement of an unknown magnetic field using one spin-j particle as a probe and another spin-j particle as an ancilla. Another example is the measurement of the magnetic field gradient between two locations probed by the two entangled spins [20, 21]. In both examples, the fact that quantum-enhanced precision can be achieved using $n \geq 3$ spin-j singlets is good news, since these states are much easier to produce than the optimal quantum states for the estimation of rotations [6, 7, 8, 17]. In addition, since three copies are sufficient to beat the classical scaling, an estimation strategy using spin-j singlets is intrinsically robust under losses.

Non-achievability of the Quantum Cramér-Rao bound in the finite-copy regime. A popular approach to quantum metrology is via the quantum Cramér-Rao bound (CRB), which lower bounds the variance with the inverse of the quantum Fisher information [23, 24, 25]. The bound is known to be achievable in the asymptotic limit where a large number of identical copies are available [26]. Practically, however, the CRB is often invoked to discuss quantum advantages in the finite-copy regime. Our result provides a strong caveat on this sort of extrapolations: for a single copy of a spin-j singlet, it implies that the variance of the three Euler angles cannot vanish with j, despite the fact that that Fisher information grows like j^2. The asymptotic achievability of the CRB is retrieved in our approach using $n \gg 1$ identical copies, with higher order corrections vanishing as $n^{3/2}$, uniformly in j. In other words, achieving the CRB requires n to be large, but not necessarily large compared to j.

Precision-enhancement from correlations in invariant degrees of freedom. In addition to activation, our results highlight another, purely quantum feature. This feature shows up already at the single-copy level: here, we know that Alice and Bob can align their axes with error $e = 4/3$—*if* they use, in addition to the spin-j singlet, a maximally entangled state between two rotationally invariant degrees of freedom. But is this resource necessary? Classically, sharing correlations between two invariant degrees of freedom is of no help for aligning directions in space. Instead, in the quantum world these correlations can make the difference: for example, for $j = 1/2$ we find that without entanglement between the invariant qubits the error has the larger value $e = 16/9$, strictly larger than $4/3$. In other words, a state that is useless for the alignment of reference frames turns out to be very useful when used in combination with other states.

In conclusion, a pair of quantum gyroscopes can be correlated in a way that makes the alignment precision independent of their size as long as a single copy is available. When two or more such pairs, the are used jointly one obtains a vanishing error. Specifically, for one copy of a spin-j singlet the error is independent of j, while for two copies the error reaches the Heisenberg scaling $1/j^2$ with probability $p \geq 43\%$ and the SQL scaling $1/j$ otherwise. Sub shot-noise scaling can be achieved with unit probability for every number of copies $n > 2$. In particular, four copies suffices to achieve the Heisenberg scaling. These results have several applications, e.g. to secret sharing of spatial directions and quantum metrology in the finite-copy regime. In addition, they shed light on the non-trivial way in which quantum resources compose in the resource theory of reference frames [28, 29, 30, 31, 32].

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