Problems of Tsirelson and Connes, and a hierarchy of combinatorial parameters

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Abstract

We develop further a general framework of combinatorial nature to study correlations. Our motives are the Tsirelson's conjecture, concerned with fundamental equivalence between the tensor product assumption and the commutativity assumption in quantum mechanics, and the Connes' embedding conjecture, concerned with operator algebras. We describe a hierarchy of parameters between the ϑ-number, which arises in information theory and semidefinite programming, and the quantum chromatic number, which arises as a value of certain non-local games. The parameters provide a novel way to study the validity of these two conjectures, allow us to give a proof of the failure of Tsirelson's original approach, and finally propose new ideas in optimization. The technical version is available at `http://arxiv.org/pdf/1407.6918v1.pdf`

In 1976, Connes casually formulated a conjecture about a fundamental approximation property for finite von Neumann algebras [1]. The conjecture has found during the years many unexpected equivalent statements. One of those appeared in the study of quantum correlations and it is linked to a seminal 1993 work of Tsirelson [2]. The statement reduces to decide whether the mathematical models of non-relativistic quantum mechanics – where observers have operators acting on a finite dimensional tensor product space – and algebraic quantum field theory – where observers have commuting operators on a single

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space, possibly infinite dimensional – produce the same set of correlations. Both, positive and negative answers to this problem have important consequences of foundational and operational type. We approach the problem from a combinatorial side, by working with graphs and their associated algebraic structures. Thus, instead of constructing the relevant sets of correlation matrices, we try to do something much more coarse by just looking for various patterns of zeroes in the sets.

The appearence of graph-theoretic parameters related to correlations goes back at least to Galliard and Wolf [3] in 2002. In the case of two-parties non-local games, one of these parameters is now called *quantum chromatic number*, χ_q; known to be loosely upper bounded by the chromatic number, χ, it has been recently employed for showing advantages of entanglement-assisted zero-error communication, and it has been proven to be intractable [4]. By denoting as $Q(n,c)$ the set of quantum correlation matrices of size $nc \times nc$, χ_q is the least c such that exists a matrix in $Q(n,c)$ with a specific pattern of zeros enforced by the rules of the non-local game. Two of us, Paulsen and Todorov [5], introduced the *commuting quantum chromatic number*, χ_{qc}, as the analogue of χ_q, but for the set of algebraic quantum field theory correlation matrices, $Q_c(n,c)$. By its statement, Tsirelson's conjecture is true if and only if $Q(n,c) = Q_c(n,c)$, for all n and c. In taking the closure of $Q(n,c)$, $Q^-(n,c)$, we can define χ_{qa}, which is an "approximate version" of χ_q. And it is known that Connes' conjecture is true if and only if $Q_c(n,c) = Q^-(n,c)$, for all n and c [6]. The integers χ_q, χ_{qc}, and χ_{qa} are an occasion to move the discussion around these problems to the realm of algebraic graph theory. Indeed, it is sufficient to find a single graph such that $\chi_{qc} \neq \chi_{qa}$ to disprove Connes. Same thing for $\chi_{qc} \neq \chi_q$ and Tsirelson. All we can say at the moment is that $\chi_{qc} \leq \chi_{qa} \leq \chi_q \leq \chi$, obviously.

However, a natural approach to show that $Q(n,c) = Q_c(n,c)$ is – roughly speaking – to start with vectors and construct commuting projections on a finite dimensional Hilbert space satisfying some orthogonality costraints that reflect the behaviour of the vectors. This was the method originally used by Tsirelson [7]. With our techniques, we are able to show that such a construction fails. The proof is based on ϑ^+, the Szegedy's variant of the ϑ-number of Lovász [8] ($\vartheta \leq \vartheta^+$): we have explicit examples such that $\vartheta^+ < \chi_{qc}$. Then, we observe that χ_{qc} is computable via semidefinite programming (SDP). The proof builds on ideas borrowed from the "NPA hierarchy" exhibited by Navascués *et al.* [9]. It uses a compactness argument to show that for the purposes of computing this integer the hierarchy terminates, but does not yield the stage at which it does so. Thus, while we can say that it is computable by one of the SDP's in the hierarchy, we cannot explicitly determine the size of this SDP. It is implausible for the SDP to be polynomial: it would disprove Connes' conjecture or go against the wide belief that P $\neq$ NP. Such a point gives further and novel evidence to the validity of the conjecture.

In the attempt of separating χ_{qc}, χ_{qa} and χ_q, we introduce some "intermediate parameters", ξ_x with $x \in \{l, q, qa, qc\}$, suggested by the notion of graph homomorphisms and generalizing the *projective rank* defined by Roberson and Mančinska [10]. We have that ξ_l ($\xi_l \leq \chi$) is the fractional chromatic number – obtainable with linear programming but intractable – , ξ_q ($\xi_q \leq \chi_q$) is the projective rank, and ξ_{qc} ($\xi_{qc} \leq \chi_{qc}$) is a new parameter that we call *tracial rank* – which is related to tracial states in C^*-algebras. Interpreting ξ_{qa} ($\xi_{qa} \leq \chi_{qa}$) remains an open problem. It is possible to notice that if there exists a graph for which ξ_q or ξ_{qc} are irrational then Tsirelson's conjecture is false. Finally, we present a new SDP that yields a parameter larger than ϑ^+ and which is still a lower bound on χ_{qc}. Independent interest on this parameter may arise from convex optimization. To determine its position in the known SDP hierarchies is an open problem.

We hope that this work will contribute to shed new light on the study of quantum correlations and to stimulate a fresh interface between combinatorics and algebra.

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