Ground state connectivity of local Hamiltonians

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Abstract

The study of ground spaces of local Hamiltonians is a fundamental task in condensed matter physics. In terms of computational complexity theory, a common focus has been to estimate a given Hamiltonian's ground state energy. However, from a physics perspective, it is often more relevant to understand the structure of the ground space itself. In this paper, we pursue this latter direction by introducing the physically well-motivated notion of "ground state connectivity" of local Hamiltonians, which captures problems in areas ranging from stabilizer codes to quantum memories. We show that determining how "connected" the ground space of a local Hamiltonian is can range from QCMA-complete to NEXP-complete. As a result, we obtain a natural QCMA-complete problem, a goal which has proven elusive since the conception of QCMA over a decade ago. Our proofs crucially rely on a new technical tool, the Traversal Lemma, which analyzes the Hilbert space a local unitary evolution must traverse under certain conditions, and which we believe may be of independent interest.

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Over the last fifteen years, the merging of condensed matter physics and computational complexity theory has given rise to a new field of study known as *quantum Hamiltonian complexity* [Osb12, GHL14]. The cornerstone of this field is arguably Kitaev's [KSV02] quantum version of the Cook-Levin theorem [Coo72, Lev73], which says that the problem of estimating the ground state energy of a local Hamiltonian is complete for the class Quantum Merlin Arthur (QMA), where QMA is a natural generalization of NP. Here, a k-local Hamiltonian is an operator $H = \sum_i H_i$ acting on n qubits, such that each local Hermitian constraint H_i acts non-trivially on k qubits. The *ground state energy* of H is simply the smallest eigenvalue of H, and the corresponding eigenspace is known as the *ground space* of H.

Kitaev's result spurred a long line of subsequent works on variants of the ground energy estimation problem (see, e.g. [Osb12, GHL14] for surveys), known as the k-local Hamiltonian problem (k-LH). For example, Oliveira and Terhal showed that LH remains QMA-complete in the physically motivated case of qubits arranged on a 2D lattice [OT08]. Bravyi and Vyalyi [BV05] proved that the *commuting* variant of 2-LH is in NP [BV05]. More recently, the complexity of 2-LH was completely characterized by Cubitt and Montanaro [CM13] in a manner analogous to Schaeffer's dichotomy theorem for Boolean satisfiability [Sch78]. Thus, k-LH has served as an excellent "benchmark" problem for delving into the complexity of problems encountered in condensed matter physics. Yet, physically speaking, what is often more relevant than the ground state energy is an understanding of the *ground space* itself. What are its properties? For example, is it topologically ordered? Can we evaluate local observables against it [Osb12]? It is this direction which we pursue in this paper.

Specifically, in this paper we define a notion of *connectivity* of the ground space of H, which roughly asks: Given ground states $|\psi\rangle$ and $|\phi\rangle$ of H as input, are they "connected" through the ground space of H? Somewhat more formally, we have (see Section 2 of technical draft for a formal definition):

Definition 1 (Ground State Connectivity (GSCON) (informal))**.** *Given as input a local Hamiltonian H and ground states $|\psi\rangle$ and $|\phi\rangle$ of H (specified via quantum circuits), as well as parameters m and l, does there exist a sequence of l-qubit unitaries $(U_i)_{i=1}^m$ such that:*

1. *($|\psi\rangle$ mapped to $|\phi\rangle$) $U_m \cdots U_1 |\psi\rangle \approx |\phi\rangle$, and*
2. *(intermediate states in ground space) $\forall\ i \in [m]$, $U_i \cdots U_1 |\psi\rangle$ is in the ground space of H?*

In other words, GSCON asks whether there exists a sequence of m unitaries, each acting on (at most) l qubits, mapping the initial state $|\psi\rangle$ to the final state $|\phi\rangle$ *through* the ground space of H. We stress that the parameters m (i.e. number of unitaries) and l (i.e. the locality of each unitary) are key; as we discuss shortly, depending on their setting, the complexity of GSCON can vary greatly.

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Motivation. The original inspiration for this work came from a recently active area in classical complexity theory on *reconfiguration* problems (see *Previous work* below for details). For example, the reconfiguration problem for 3SAT asks: Given a 3SAT formula ϕ and satisfying assignments x and y for ϕ, does there exist a sequence of bit flips mapping x to y, such that each intermediate assignment encountered is also a satisfying assignment for ϕ? Although the classical study of reconfiguration problems is arguably mostly interesting from a theoretical perspective, its quantum variant (i.e. GSCON) turns out to be physically very relevant. We now discuss connections to *quantum memories* and *stabilizer codes*.

Quantum memories. A key challenge in building quantum computers is the implementation of long-lived qubit systems. In low-temperature condensed matter systems, one approach is to encode a qubit in the ground state of a gapped Hamiltonian with a degenerate ground space. Here, the degeneracy ensures the qubit has at least two basis states, logical $|\widetilde{0}\rangle$ and $|\widetilde{1}\rangle$, and the gap ensures that external noise does not (easily) take a ground state out of the ground space. However, this is not sufficient — although environmental noise may not take the state *out* of the ground space, it can still alter the state *within* the ground space (e.g. inadvertently map $|\widetilde{0}\rangle$ to $|\widetilde{1}\rangle$). Thus, making the typical assumption that errors act locally, it should ideally not be possible for $|\widetilde{0}\rangle$ to be mapped to $|\widetilde{1}\rangle$ through the ground space via a sequence of local operations. This is precisely the principle behind Kitaev's toy chain model [Kit01], and the motivation behind the toric code [Kit03] (see also [KL09]). This notion of how "robust" a quantum memory is can thus be phrased as an instance of GSCON: Given a gapped Hamiltonian H, a ground state $|\psi\rangle$ to which the quantum memory is initialized, and an undesired ground state $|\phi\rangle$, is there a sequence of local errors mapping the state of our quantum memory through the ground space from $|\psi\rangle$ to $|\phi\rangle$?

Stabilizer codes. Roughly, a stabilizer code [Got97] is a quantum error-correcting code defined by a set of commuting Hermitian operators, $S = \{ G_1, \ldots, G_k \}$, such that $G_i \neq -I$ and $\|G_i\|_\infty \leq 1$ for all $G_i \in S$. The *codespace* for S is the set of all $|\psi\rangle$ satisfying $G_i |\psi\rangle = |\psi\rangle$ for all $i \in [k]$. In other words, defining G_i^+ as the projection onto the $+1$ eigenspace of G_i, the codespace is the ground space of the positive semidefinite Hamiltonian $H := \sum_{i=1}^{k}(I - G_i^+)$. Typically, errors are assumed to occur on a small number of qubits at a time; with this assumption in place, the following is a special case of GSCON: Given H and codewords $|\psi\rangle$ and $|\phi\rangle$, does there exist a sequence of at most m local errors mapping $|\psi\rangle$ to $|\phi\rangle$, such that the entire error process is undetectable, i.e. each intermediate state remains in the codespace?

Results. Having motivated GSCON, we now informally state our results.

Theorem 2 (See Theorem 5.1 of technical draft for a formal statement)**.** GSCON *for polynomially large m (i.e. for polynomially many local unitaries U) and $l = 2$ (i.e. 2-qubit unitaries) is* QCMA*-complete.*

Here, QCMA is QMA except with a classical prover [AN02]. Theorem 2 says that determining whether there exists a poly-size quantum circuit mapping $|\psi\rangle$ to $|\phi\rangle$ through the ground space of H is QCMA-complete.

Theorem 3 (See Theorem 6.1 of technical draft for a formal statement)**.** GSCON *for exponentially large m (i.e. for exponentially many local unitaries U) and $l = 1$ (i.e. 1-qubit unitaries) is* PSPACE*-complete.*

Theorem 3 says that determining whether there exists an exponential length sequence of 1-qubit unitaries mapping $|\psi\rangle$ to $|\phi\rangle$ through the ground space of H is PSPACE-complete. Note that the settings of both m and l above are crucial for our proofs; for example, for exponential m and $l = 2$ (i.e. 2-qubit unitaries instead of 1-qubit unitaries), our proof of containment in PSPACE for Theorem 3 does not hold.

Finally, we consider a succinct variant of GSCON, called SUCCINCT GSCON, in which the Hamiltonian H has a succinct circuit description, and the initial and final states $|\psi\rangle$ and $|\phi\rangle$ are product states.

Theorem 4 (See Theorem 7.4 of technical draft for a formal statement)**.** SUCCINCT GSCON *for exponentially large m (i.e. for exponentially many local unitaries U) and $l = 1$ (i.e. 1-qubit unitaries) is* NEXP*-complete.*

Proof techniques. Our results crucially rely on a new technical lemma called the Traversal Lemma, as well as the use of ϵ-nets and what we call ϵ-pseudo-nets. We now discuss the techniques behind Theorem 2 (QCMA-completeness) in more detail, as they perhaps best exemplify the approaches taken in this work.

We begin by outlining the central idea behind the construction in our QCMA-hardness proof. Let V be an arbitrary QCMA verification circuit, and let H' be the local Hamiltonian obtained from V via Kitaev's circuit-to-Hamiltonian construction [KSV02]. Then, we design the input Hamiltonian H to GSCON so that "traversing its ground space" is equivalent to simulating the following protocol: Starting from the all-zeroes state, prepare the ground state of H' (which can be done efficiently since V is a QCMA circuit), and subsequently flip a set of special qubits called *GO* qubits. This latter step "activates" the check Hamiltonian H, which now "verifies" that the ground state prepared is indeed correct. Finally, uncompute the ground state to arrive at a target state of all-zeroes except in the *GO* register, which is now set to all ones.

To prove correctness of this construction, our main technical tool is a new lemma we call the *Traversal Lemma* (Lemma 4.2 in technical draft), which analyzes the Hilbert space a local unitary evolution must traverse in certain settings. Specifically, define two states $|\psi\rangle$ and $|\phi\rangle$ as *k-orthogonal* if for any k-local unitary U, we have $\langle\phi| U |\psi\rangle = 0$. In other words, any application of a k-local unitary leaves $|\psi\rangle$ and $|\phi\rangle$ orthogonal. Then, the Traversal Lemma says that for k-orthogonal states $|\psi\rangle$ and $|\phi\rangle$, if we wish to map $|\psi\rangle$ to $|\phi\rangle$ via a sequence of k-local unitaries, then at some step in this evolution we must leave the space spanned by $|\psi\rangle$ and $|\phi\rangle$, i.e. we must have "large" overlap with the orthogonal complement of $|\psi\rangle$ and $|\phi\rangle$. To prove the Traversal Lemma, we use a combination of the Gentle Measurement Lemma of Winter [Win99] and an idea inspired by the quantum Zeno effect.

Finally, to show containment in QCMA, we introduce the notion of *ϵ-pseudo-nets*, which allow us to easily discretize the space of d-dimensional unitary operators for any $d \geq 2$. Such pseudo-nets come with a tradeoff: On the negative side, they contain non-unitary operators. On the positive side, they are not only straightforward to construct, but more importantly, they have the following property: Given any element A in the pseudo-net, there are efficient *explicit* protocols for checking if A is close to unitary, and if so, for "rounding" it to such a unitary.

Previous work. To the best of our knowledge, our work is the first to study reconfiguration in the quantum setting. In contrast, in the classical setting, such problems have recently received much attention. In particular, our work was inspired by the paper of Gopalan, Kolaitis, Maneva, and Papadimitriou [GKMP06, Sch13], which shows that determining whether two solutions x and y of a Boolean formula are connected through the solution space is either in P or is PSPACE-complete, depending on the constraint types allowed in the formula. More recently, Mouawad, Nishimura, Pathak and Raman [MNPR14] studied the variant of this problem in which one seeks the *shortest* possible Boolean reconfiguration path; they show this problem is either in P, NP-complete, or PSPACE-complete. In this sense, our definition of GSCON can be thought of as a quantum generalization of the problem studied in Reference [MNPR14]. More generally, since the work of Reference [GKMP06], a flurry of papers have appeared studying reconfiguration for problems ranging from Boolean satisfiability to vertex cover to graph coloring [CvdHJ08, BC09, BJL$^+$11, CvdHJ11, FHHH11, IDH$^+$11, Bon12, IKD12, IKOZ12, KMM12, Sch13, BB13, MNR$^+$13, MNPR14, MNR14].

Significance. We have discussed GSCON in terms of physical motivation (see connections to stabilizer codes and quantum memories discussed in *Motivation* above). Let us now discuss its appeal from a complexity-theoretic perspective, in particular with regard to the class QCMA. It has been over a decade since the introduction of QCMA by Aharonov and Naveh [AN02] and since that time a handful of complete problems have been discovered for it [WJB03, WY08, JW06, GK12], such as determining if a quantum circuit acts almost as the identity on computational basis states [WJB03] and minimizing the Hamming weight of a string accepted by a certain class of quantum circuits [GK12]. However, in contrast to the canonical QMA-complete local Hamiltonian problem, the known QCMA-complete problems are arguably not very natural. To this end, our work reveals the first physically well-motivated QCMA-complete problem, filling this decade-long open gap.

As for our proof techniques, we believe the Traversal Lemma may prove useful in its own right. For example, in quantum adiabatic algorithms, it is often notoriously difficult to understand how a quantum state evolves in time from an easy-to-prepare initial state to some desired final state. The Traversal Lemma gives us a tool for studying the behaviour of such evolutions, playing a crucial role in our analysis here. We remark, however, that in quantum adiabatic evolution, the Hamiltonian itself changes with time, whereas here our Hamiltonian is fixed and we apply local unitary gates to our quantum state.

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