Excluding one of the parties and Markovianizing of tripartite quantum states

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We consider a distributed decoupling task for a tripartite system in which Bob and Charlie "exclude" Alice by destroying correlation between BC and A in a tripartite state ρ^{ABC} shared by the parties using only local random operations. We refer to this task as *excluding*. In an asymptotic limit of infinite copies of ρ^{ABC}, we investigate the *excluding cost*, namely, the cost of randomness required to accomplish excluding. In particular, we consider a case where the initial state is one obtained from a bipartite unitary U as $\Psi(U)^{ABC} = (U^{AB} \otimes I^C)(I^A/d \otimes |\Phi_d\rangle\langle\Phi_d|^{BC})(U^{\dagger AB} \otimes I^C)$, for which excluding can be accomplished by only Charlie's local random operation. We prove that the excluding cost in this case is equal to the cost of randomness required for turning a tripartite pure state sufficiently close to a quantum Markov chain by a random operation, in the asymptotic limit of infinite copies. We derive the single-letter formula of this "Markovianizing cost" for arbitrary tripartite pure state by using the data compression theorem for mixed-state quantum information sources. We show that the excluding cost derived here gives an upper bound on the cost of resources required in a model of distributed quantum computation, and that it gives the optimal quantum communication rate required in distributed compression of tripartite quantum state in a particular setting.

1 Overview

The decoupling approach has played a significant role in the field of quantum information theory for a decade [1–8]. The approach was first established by [1–3] in an asymptotic regime, and it was shown that most of the central coding theorems in quantum Shannon theory are systematically derived from protocols known as quantum state merging [2] and the fully-quantum Slepian Wolf [3]. Recent studies also develop the decoupling approach in the single-shot scenario [4, 5]. The main concern in this approach is the minimum amount of randomness required to destroy correlation between two quantum systems, which is revealed by the so-called "decoupling theorem". Depending on types of operations applied to destroy the correlation, there are several formulations of the decoupling theorem, such as one based on the partial trace [3, 4], random unitary operations [6], projective measurements [1, 2], and arbitrary CPTP maps [5]. However, most of the studies so far has only focused on the bipartite setting. This is problematic particularly if we consider multipartite quantum communication tasks [9–13].

In this contribution, we introduce a task that we call *excluding*, which is a generalization of bipartite decoupling to more than bipartite scenario. Here, Alice, Bob and Charlie initially share copies of a state $(\rho^{ABC})^{\otimes n}$. The task is for Bob and Charlie to destroy correlation between BC and A by local random unitary operations on B^n and C^n, respectively. In the asymptotic limit of $n \to \infty$, and under the condition that the excluding is accomplished within a vanishingly small error, we investigate the minimum number of random unitaries per copy necessary to apply for Bob and Charlie.

Definition 1.1 We say that the rate pair (R_1, R_2) is achievable in excluding of A with respect to ρ^{ABC} if, for any $\epsilon > 0$ and for sufficiently large n, there exists a random unitary map $\mathcal{T}_1^{B^n} : \tau \mapsto 2^{-nR_1} \sum_{k=1}^{2^{nR_1}} V_k \tau V_k^\dagger$ on B^n and $\mathcal{T}_2^{C^n} : \tau' \mapsto 2^{-nR_2} \sum_{l=1}^{2^{nR_2}} W_l \tau' W_l^\dagger$ on C^n such that

$$\left\| (\mathrm{id}^{A^n} \otimes \mathcal{T}_1^{B^n} \otimes \mathcal{T}_2^{C^n})((\rho^{ABC})^{\otimes n}) - (\rho^A)^{\otimes n} \otimes (\mathcal{T}_1^{B^n} \otimes \mathcal{T}_2^{C^n})((\rho^{BC})^{\otimes n}) \right\|_1 \leq \epsilon.$$

We consider a case where ρ^{ABC} is a particular mixed state $\Psi(U)^{ABC} = (U^{AB} \otimes I^C)(I^A/d \otimes |\Phi_d\rangle\langle\Phi_d|^{BC})(U^{\dagger AB} \otimes I^C)$, $|\Phi_d\rangle$ is a maximally mixed state and U^{AB} is a bipartite unitary, and d is the dimension of the local systems. In this case, excluding can be accomplished by only random operations on Charlie's system, since $\Psi(U)^{AB} = I^A/d \otimes I^B/d$ and thus complete randomization of Alice's system turns the state into $\Psi(U)'^{ABC} = I^A/d \otimes I^B/d \otimes I^C/d$ at the cost of $2\log d$ bits of randomness per copy. Thus the rate pair $(R_1 = 0, R_2 = 2\log d)$ is achievable. However, this strategy may cost too much randomness for excluding, because it decorrelates B and C in addition to destroy correlation between BC and A. There might be a more efficient protocol which requires less randomness by selectively destroying correlation between BC and A, without decorrelating B and C. It turns out to be impossible to show the existence of such an efficient strategy by a straightforward application of random coding method using the Haar distributed unitary ensemble. In this contribution, we prove the existence of an optimal strategy for excluding $\Psi(U)^{ABC}$ by constructing a random coding method based on the structure of a quantum Markov chain. We derive, as a function of U, the excluding cost of U defined as $\mathrm{Exc}(U) := \inf\{R \mid (R_1 = 0, R_2 = R) \text{is achievable in excluding of } A \text{ with respect to } \Psi(U)^{ABC}\}$.

2 Results

It is proved in [14] that, associated with any bipartite state Ψ^{AC}, there exists an essentially unique unitary isomorphism $\Gamma : C \to c_0 c_L c_R$ that satisfies the following two conditions. An algorithm for obtaining Γ is given in [14, 15].

1. Ψ^{AC} is decomposed as

$$\Gamma^C \Psi^{AC} \Gamma^{\dagger C} = \sum_{j\in J} p_j |j\rangle\langle j|^{c_0} \otimes \varphi_j^{Ac_L} \otimes \omega_j^{c_R}, \tag{2.1}$$

   where $\{p_j\}_{j\in J}$ is a probability distribution, $\varphi_j \in \mathcal{S}(\mathcal{H}^A \otimes \mathcal{H}^{c_L})$, $\omega_j^{c_R} \in \mathcal{S}(\mathcal{H}^{c_R})$, and $\langle j|j'\rangle = \delta_{jj'}$.

2. Any quantum operation on C that leaves Ψ^{AC} invariant has a Stinespring dilation of the form $\mathcal{E}(\rho) = \mathrm{Tr}[U^{CE}(\rho^C \otimes |0\rangle\langle 0|^E) U^{\dagger CE}]$, where U^{CE} is a unitary that is decomposed as $\Gamma^C U^{CE} \Gamma^{\dagger C} = \sum_{j\in J} |j\rangle\langle j|^{c_0} \otimes I^{c_L} \otimes U_j^{c_R E}$, and $U_j^{c_R E}$ are unitaries that satisfy $\mathrm{Tr}_E[U_j(\omega_j^{c_R} \otimes |0\rangle\langle 0|^E) U_j^\dagger] = \omega_j^{a_R}$ for all j.

A tripartite state Υ^{ABC} is called a quantum Markov chain conditioned by B if it satisfies $I(A : C|B) = 0$ [14]. Our main results are summarized in the following statements.

Definition 2.1 We say that a tripartite state Ψ^{ABC} is turned to a Markov state conditioned by B with the randomness cost R on C if, for any $\epsilon > 0$ and for sufficiently large n, there exist a random unitary operation $\mathcal{T}_n : \tau \mapsto 2^{-nR} \sum_{k=1}^{2^{nR}} V_k \tau V_k^\dagger$ on C^n and a quantum Markov chain $\Upsilon^{A^n B^n C^n}$ conditioned by B^n such that $\|\mathcal{T}_n(\Psi^{\otimes n}) - \Upsilon^{A^n B^n C^n}\|_1 \leq \epsilon$. The "Markovianizing cost" is defined as $M_{C|B}(\Psi^{ABC}) := \inf\{R \mid \Psi^{ABC}$ is turned to a Markov state conditioned by B with the randomness cost R on $C\}$.

Theorem 2.2 Let $|\Psi\rangle^{ABC}$ be a pure state such that Ψ^{AC} is decomposed as Eq.(2.1). We have

$$M_{C|B}(\Psi^{ABC}) = H(\{p_j\}_{j\in J}) + 2\sum_{j\in J} p_j S(\varphi_j^{c_L}).$$

Theorem 2.3 Consider a tripartite state $|\Psi(U)\rangle^{\tilde{A}BC} := (U^{AB} \otimes I^{A'C})|\Phi_d\rangle^{AA'}|\Phi_d\rangle^{BC}$, where $\tilde{A} = AA'$. Define *Markovianizing cost of* U as $M(U) := M_{C|B}(\Psi(U)^{\tilde{A}BC}$. Then we have $\mathrm{Exc}(U) = M(U)$.

In the optimality proof of Thm.2.2, we use the data compression theorem for mixed-state quantum information sources [16]. In the achievability proof, we construct a random coding method using the ensemble of unitaries of the form $V = \sum_{j\in J} |j\rangle\langle j|^{c_0} \otimes v_j^{c_L} \otimes I^{c_R}$, where we choose each v_j randomly and independently according to the Haar measure. The same method is used to construct a random unitary operation that "selectively decorrelates AB and C" in the achievability proof of Thm.2.3. The optimality proof is based on a recently proposed characterization of a quantum Markov chain in terms of commutator [17]. The two theorems show that neither the Markovianizing cost of Ψ nor the excluding cost of U is a continuous function of its argument.

3 Applications

In our recent paper [18], we formulated a problem of reducing the entanglement cost and the classical communication cost in an LOCC implementation of bipartite unitaries in an asymptotic scenario. Combined with the results obtained there, we prove that the Markovianizing cost $M_{A|B}(\Psi(U))$ gives an upper bound on the cost of entanglement, forward and backward classical communication required therein. We strongly expect that it is also optimal in arbitrary two-round protocols. More precisely, we prove the following statement.

Definition 3.1 Consider a unitary operator $U : \mathcal{H}^A \otimes \mathcal{H}^B \to \mathcal{H}^A \otimes \mathcal{H}^B$ acting on two d-level systems A and B. Let $|\Phi(U)\rangle := (U^{AB} \otimes I^{R_A R_B})|\Phi_d\rangle^{AR_A}|\Phi_d\rangle^{BR_B}$ where Φ_d is a d-dimensional maximally entangled state. Let Alice and Bob have registers A_0, A_1 and B_0, B_1, respectively. We refer to the following quantum operation $\mathcal{M}_n$ as an entanglement-assisted LOCC implementation of $U^{\otimes n}$ with the error ϵ_n, the entanglement cost $\log K_n - \log L_n$, the forward communication cost $C_n^{\rightarrow}$, and the backward communication cost $C_n^{\leftarrow}$. Here, $\mathcal{M}_n : A^n A_0 \otimes B^n B_0 \to A^n A_1 \otimes B^n B_1$ is an LOCC and

$$F(\rho(\mathcal{M}_n), \Phi(U)^{\otimes n} \otimes \Phi_{L_n}^{A_1 B_1}) \geq 1 - \epsilon_n$$

for $\rho(\mathcal{M}_n) = (\mathcal{M}_n \otimes \mathrm{id}^{R_A R_B})(|\Phi_d^{AR_A}\rangle^{\otimes n}|\Phi_d^{BR_B}\rangle^{\otimes n}|\Phi_{K_n}\rangle^{A_0 B_0})$. $C_n^{\rightarrow}$ and $C_n^{\leftarrow}$ is the total amount of classical communication transmitted from Alice to Bob and Bob to Alice, respectively, in $\mathcal{M}_n$. A rate triplet $(R, C^{\rightarrow}, C^{\leftarrow})$ is said to be achievable if there exists a sequence of entanglement-assisted LOCC implementations of $U^{\otimes n}$ such that $\epsilon_n \to 0$, $\frac{1}{n}(\log K_n - \log L_n) \to R$, $\frac{1}{n}C_n^{\rightarrow} \to C^{\rightarrow}$ and $\frac{1}{n}C_n^{\leftarrow} \to C^{\leftarrow}$ in the limit of $n \to \infty$.

Theorem 3.2 A rate triplet $(R, C^{\rightarrow}, C^{\leftarrow})$ is achievable if $R, C^{\rightarrow}, C^{\leftarrow} > M(U)$.

We also apply our results to an analysis of distributed compression of quantum states. In the m-party distributed compression, senders $A_1, \cdots, A_m$ initially share n identical copies of a state $|\psi\rangle^{A_1 \cdots A_m R}$ with an inaccessible reference system R. The task is to compress their shares and transmit them to a receiver with a vanishingly small error, by using as small amount of quantum communication from each sender to the receiver as possible, but with no communication among senders. We are interested in finding out the set of the achievable quantum communication rates $(Q_1, \cdots, Q_m)$ from each senders to the receiver. Contrary to the bipartite setting [3, 9], little has been known on more than bipartite cases [10].

We consider distributed compression of a tripartite quantum state $\tilde{\Psi}(U)^{\tilde{A}\tilde{B}C} := \Psi(U)^{ABC} \otimes \Phi_{d^3}^{A'B'}$, where $\tilde{B} = BB'$, $\tilde{C} = CC'$ and Φ_{d^3} is the maximally entangled state with the Schmidt rank d^3. In a particular situation in which $Q_C = 0$ and $Q_{\tilde{A}}$ can be arbitrarily large, we can evaluate the minimum achievable rate $Q_{\tilde{B}}$. We prove the following.

Theorem 3.3 The rate $Q_{\tilde{B}} = \frac{1}{2}R$ is achievable if and only if $R \geq M(U)$.

4 Concluding Remarks

We formulate and investigate a multipartite version of distributed decoupling to exclude one of the parties in an asymptotic scenario. By using the structure of a quantum Markov chain, we construct a random coding method novel enough to derive the optimal cost of randomness required in excluding. Our results opens a new possibility to investigate multipartite quantum communication tasks, as well as to analyze the structure of multipartite quantum correlations, from the decoupling point of view.

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