An invariant of topologically ordered states under local unitary transformations

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1. Background

Gapped quantum Hamiltonians are commonly said to represent the same quantum phases of matter, if there exists a gapped one-parameter family of Hamiltonians $H(s)$ that connects them. Here, the gap refers to the energy gap separating the ground state from the rest of the spectrum. The ground state is distinguished throughout the Hamiltonian path $H(s)$, yet the high energy states can be mixed. This definition of equivalence is motivated by our general interest in low energy phenomena where quantum fluctuation is expected to be large.

It is important that properties of the ground state do not change drastically along the Hamiltonian path. This is mathematically supported by quasi-adiabatic continuation [1], which says that the instantaneous ground state $|\psi(s)\rangle$ of $H(s)$ can be well approximated as $|\psi(s)\rangle \simeq U(s)|\psi(0)\rangle$ by a quantum circuit $U(s)$ of depth that is much smaller than the system size, where $0 \le s \le 1$. Since a quantum circuit of small depth maps any local observable to local observable, any correlation in the ground state will not be changed much at long distances. Conversely, given a quantum circuit $V = V_d V_{d-1} \cdots V_1$ of small depth d one can devise a conntinuous gapped Hamiltonian path defined by $H(s) = V(s) H V(s)^\dagger$ where $V(n/d)$ implements the first n layers of V for $n = 1, 2, \ldots, d$.

This strongly suggests that all relevant information of the quantum phase of matter is contained in a ground state vector up to local unitary transformations. From an information-theoretic point of view, this means that a quantum phase of matter is essentially an entanglement pattern; recall that the any bipartite entanglement measure such as entanglement entropy is required to be invariant under any basis change on either party. One can reversely say that the bipartite entanglement is what remains invariant under such local basis change. Analogously, if we are interested in the many-body quantum state and its entanglement, any local unitary transformation should be allowed, and the requirement of the transitivity for an equivalence relation drives us to consider quantum circuits of finite depth when studying the entanglement of many-body quantum state. Thus, we say different entanglement patterns represent distinct quantum phases of matter, and for the lack of any symmetry restriction one may call them topological orders.

As the bipartite entanglement is quantified by the Schmidt coefficients, which are invariants under either party's basis change, it is natural and important to look for invariants of quantum state vectors under local quantum circuits. Various quantities and properties have been proposed as invariants of quantum states [2, 3, 4, 5]. The local indistinguishability is perhaps the easiest to define mathematically, though hard to grasp intuitively. It is the property of a quantum state that it has an orthogonal partner state which has the identical reduced density matrix for every local region. Any local observable that distinguishes the pair, if exists, will remain local under an arbitrary small-depth quantum circuit. Thus, the local indistinguishability is an invariant of the quantum state. It is binary valued, true or false, and hence is crude.

A theoretical guiding question to judge the strength of an invariant would be the following: How deep a quantum circuit must be in order to transform a state to another? A good invariant should be able to distinguish states that are intuitively thought to be different, and should show that any transformation between the states takes a quantum circuit whose depth is increasing with the system size. A simple problematic example is the toric code state on a sphere. It encodes zero qubit in the ground state subspace; there is exactly one ground state. The local indistinguishability cannot be used to differentiate it from the trivial product state [6]; we need a finer invariant.

2. Result

In this paper, we define a class of quantum states and an associated matrix $\tilde{S}$, and prove that $\tilde{S}$ is an invariant under small-depth quantum circuits. The matrix $\tilde{S}$ is motivated by and defined analogously with the so-called topological S-matrix of anyon theories [5, 7]. We define $\tilde{S}$ using a Hamiltonian, but prove that it is in fact independent of the Hamiltonian; if two Hamiltonians H_1, H_2 have a ground state $|\psi\rangle$ in common, then $\tilde{S}(H_1) = \tilde{S}(H_2)$. In this sense, our $\tilde{S}$-matrix is a quantity of the state. Moreover, $\tilde{S}$ is not affected at all by inserting or removing trivial ancilla qudits. By investigating situations where $\tilde{S}$ can be consistently defined as we deform the state by quantum circuits, we conclude that any transformation between quantum states with distinct $\tilde{S}$-matrices must be a quantum circuit of depth that is at least linear in the system's diameter. The trivial product state and the toric code state on the sphere have different $\tilde{S}$-matrices, proving that our invariant is finer than the mentioned local indistinguishability.

The class of states we are considering are ground states of local commuting projector Hamiltonians with two extra conditions. The number of spatial dimensions is conventionally restricted to 2, but it can be naturally generalized. To define the $\tilde{S}$-matrix, we first consider particle type projectors. They act on an annulus of the lattice, projecting onto states with a definite topological charge supported in the disk that the annulus encloses. A topological charge is by definition a set of states that are connected by arbitrary local operator acting on a local region surrounded by the annulus, which serves a role to separate the charge with other possible excitations. The particle type projectors are then the central elements of the algebra of operators that leave the annulus free of excitations. More precisely, let $\mathcal{A}$ be the set of all operators supported on an annulus that commutes with every term h_a in the Hamiltonian, and $\mathcal{N}$ be the set of all operators $O \in \mathcal{A}$ such that $O\left(\prod h_a\right) = 0$, where the product runs over terms h_a whose support overlaps with the annulus. The particle type projectors are the central canonical projectors of the C^*-algebra $\mathcal{A}/\mathcal{N}$, supplied by the Artin-Wedderburn theorem. For the toric code, these particle type projectors are precisely $\frac{1}{4}(I \pm \bar{X})(I \pm \bar{Z})$, where $\bar{X}$ and $\bar{Z}$ are the usual string operators of Pauli matrices. The $\tilde{S}$-matrix

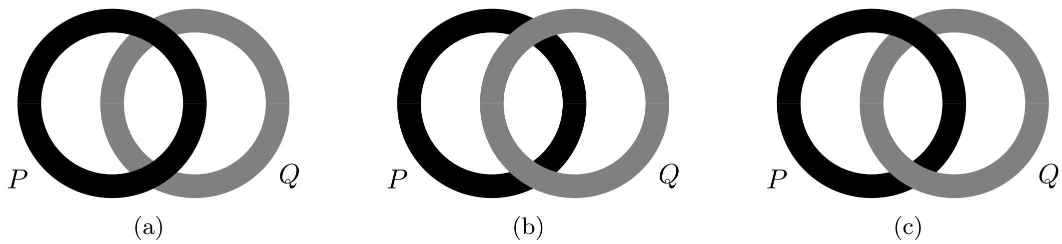

Figure 1: Two annuli of the same radius on a large plane are intersecting at two diamond-like regions. The distance between the two diamonds are comparable with the radius of the annuli. P and Q denote operators supported on the left and right annulus, respectively. In (a), the usual product $P \cdot Q$ is depicted. The operator on the left is drawn closer to the reader. In (b), the product $Q \cdot P$ is shown. In (c), the twist product $P \infty Q$ is depicted. The order of the multiplication is reversed for operator components in the bottom region. $\tilde{S}_{ab}$ is defined by $\langle\psi| \pi_a \infty \pi_b |\psi\rangle$ where π_a and π_b are particle type projectors on the left and right annulus, respectively.

is defined by the expectation value of these particle type projectors after a special *twist* product. See Fig. 1.

The two extra conditions for the Hamiltonians are as follows. The first one is so-called local topological order condition, stating that Hamiltonian terms that act on a small disk should uncover the reduced density matrix of the global ground state for the disk. This condition is elaborated in the gap stability proof of topological order [8]. The second one is a new one, which we call stable logical algebra condition, stating that the *logical algebra* $\mathcal{A}/\mathcal{N}$ does not depend on the thickness of the annulus. This seems to be closely related to the condition that there are finitely many particle types.

3. Overview of the proof

We establish isomorphisms for logical algebras $\mathcal{C}(H) := \mathcal{A}(H)/\mathcal{N}(H)$, where H is a Hamiltonian satisfying our two conditions. The invariance of $\tilde{S}$ under quantum circuits W follows from $\mathcal{C}(H) \cong \mathcal{C}(WHW^\dagger)$, which is rather straightforward. More involved is to establish $\mathcal{C}(H_1) \cong \mathcal{C}(H_2)$ when $H_{1,2}$ share a ground state. To this end, we introduce a manifestly Hamiltonian-independent notion of *locally invisible* operators. We say an operator O is locally invisible if for any state $|\phi\rangle$ that has the same reduced density matrix on a disk D with the global ground state $|\psi\rangle$, $O|\phi\rangle$ has the same reduced density matrix on the "interior" of the disk D with $|\psi\rangle$. That is, the action of O cannot be locally detected. By factoring out a set $\mathcal{M}$ of null operators that annihilate local reduced density matrices, from the set $\mathcal{I}$ of all locally invisible operators on an annulus, we find a sequence of maps $\mathcal{C}(H) \xrightarrow{\bar{\iota}} \mathcal{I}/\mathcal{M} \xrightarrow{\bar{\phi}} \mathcal{C}(H)$. Note that $\mathcal{I}/\mathcal{M}$ is not an algebra in general because the local invisibility is not a linear condition. Nevertheless, $\bar{\iota}$ and $\bar{\phi}$ are inverses of each other, and the composition $\bar{\phi} \circ \bar{\iota}$ preserves addition and multiplication. We thus obtain an algebra-isomorphism $\mathcal{C}(H_1) \to \mathcal{C}(H_2)$.

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