

Single-qubit gate protection in a symmetry-protected topological phase

Abhishodh Prakash and Tzu-Chieh Wei

*C. N. Yang Institute for Theoretical Physics and Department of Physics and Astronomy,
State University of New York at Stony Brook, Stony Brook, NY 11794-3840, USA*

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I. MOTIVATION OF OUR WORK

Symmetry-protected topological (SPT) phases have topological order that is not characterized by a local order parameter and their existence requires symmetry to be preserved [1–5]. It one of the research frontiers in classifying topological phases in condensed matter physics. Ground states of SPT phases cannot continuously connect to trivial product states without either closing the gap or breaking the protected symmetry. Their entanglement is short-ranged, as opposed to intrinsic topological order that does not require symmetry to stabilize it. As is well-known, some intrinsic topological phase can be used for topological quantum computation. Is there a link between the short-ranged entangled SPT orders and quantum computation?

In a seminal paper by Else et al. [6] it was shown that some of these SPT ground states can serve as resource states for realizing certain gate operations in quantum computation by local measurement. In particular, in the so-called Haldane phase, protected by $Z_2 \times Z_2$ symmetry, the *identity gate* is shown to be naturally protected. The protection of *arbitrary* single-qubit gates does not seem to be generic, as it is only possible at the so-called Affleck-Kennedy-Lieb-Tasaki (AKLT) point in the Haldane phase. Even by imposing higher symmetry such as the full $SO(3)$ symmetry, it was not possible to constrain the ground state so as to protect arbitrary gates. A question that one wishes for an affirmative answer is “Could there be a symmetry-protected topological phase where arbitrary single-qubit gates are protected?”

Of course, an even more desirable breakthrough would be to find a SPT phase where all universal gates are protected. But this requires much more profound understanding of the 2D SPTO and a much more thorough (if not complete) classification of 2D universal resources in measurement-based quantum computation (MBQC), currently missing.

II. OUR MAIN RESULTS

The results by Else et al. [6] hinge on features of specific Abelian groups, i.e., groups whose projective representation possesses a maximally noncommutative factor system. *In our work we develop a formalism that allows us to treat an arbitrary finite group G , either Abelian or non-Abelian, so that we can examine the associated SPT phases and protected gate operations.* The merit of this work is the general formulation of a Wigner-Eckart-like theorem for discrete groups.

(1) Using our formalism, we can reproduce the results in Ref. [6] on the spin-1 system with $Z_2 \otimes Z_2$ symmetry. (2) As an application, we find that a 1D topologically non-trivial SPT phase associated to the symmetry group A_4 (the alternating group of degree 4) acting on a three-dimensional on-site irreducible representation (i.e., physical spin-1 entities) also protects the identity gate operation. (3) As opposed to our previous claim, further imposing either inversion or time-reversal symmetry does not lead to unique ground state as the AKLT [7]. (4) The weaker statement is that somehow within the A_4 symmetry imposed there exists an extended region inside but not the entire A_4 phase that the AKLT state is the unique ground state. Hence only in this sub-region of the A_4 phase, the arbitrary one-qubit gate is protected. Therefore, the natural gate protection still requires some fine tuning of system parameters (even under symmetry constraint). For this claim, we were able to construct a Hamiltonian to realize the nontrivial A_4 SPT phase and check numerically that

under this particular Hamiltonian there indeed exists an extended region, where the ground state is uniquely the AKLT state when \mathbb{A}_4 and inversion symmetry is preserved in the entire symmetric region. But adding a biquadratic term (which is $SO(3)$ symmetric) to the Hamiltonian distorts the ground state away from the AKLT state, and hence, the arbitrary single-qubit gates are not naturally protected, with using further buffering scheme proposed with Miller and Miyake.

Significance. Our formalism can deal with arbitrary on-site discrete symmetry. While we still does not manage to find an entire SPT phase that protects arbitrary single-qubit operations, the region of protected single-qubit operations can in some sense enlarged by enlarging symmetry, such as from $Z_2 \times Z_2$ to $\mathbb{A}_4 + \text{inversion}$. While single-qubit operations are known to not be universal and full quantum computational universality would requires resource states of higher spatial dimensions, we believe that our results shed light on the possibility or at least motivation to go after the SPT phases where all universal gates might be protected. Moreover, our results connect to the recent develop of classification of topological phases of matter in condensed matter physics, and our formalism may have its own application in that respect.

Matrix product states and measurement-based quantum computation. The measurement-based quantum computation (MBQC) is a quantum computational scheme that makes use of only local measurements on a suitably entangled resource state [8]. In this work we focus on one-dimensional resource states, and the matrix-product-state (MPS) representation [9, 10] is particular useful for the novel MQBC scheme [11, 12], usually written as

$$|\psi\rangle = \sum_{i_1 \dots i_N} \langle R | A_N^{i_N} \dots A_2^{i_2} A_1^{i_1} | L \rangle |i_1\rangle \dots |i_N\rangle \quad (1)$$

Here the vectors $|L\rangle$ and $|R\rangle$ live in the virtual space and encode the boundary condition for the finite chain. Suppose a projective measurement of the 1^{st} spin yields the outcome being a projection onto state $|\phi'\rangle \in \{|\phi_i\rangle\}$, the resulting a gate $A'_1 = \sum_{i_1} \langle \phi' | i_1 \rangle A_1^{i_1}$ acting on vector $|L\rangle$, hence gate operations. Therefore, the key is to identify A_i 's.

New formalism. We refer the detail derivation to our technical paper, attached with this submission. The upshot is that using symmetry consideration and tensor decomposition, we can determine the structure of the ground-state wavefunction in the symmetry-protected topological phase via the MPS. The wavefunction is organized into two parts: (1) the protected part, whose form is inferred from the symmetry and (2) the so-called junk part, whose form is not constrained by the symmetry. A convenient labeling of the element in the vector space (of the indices of the matrices and physical dimensions) is $|i\rangle = |a_i, m_i, d_i\rangle$ where a_i labels the irrep. of group G and is analogous to the angular momentum label in $SU(2)$; m_i labels the state in a_i and is analogous to the azimuthal quantum number m_i ; d_i labels which copy of the irreducible representation a_i is being considered. Symmetry transformations act on the m_i labels of each sector a_i but leave the d_i labels alone. Using Schur's lemma we arrive at the following structure for the matrices

$$A_{(a_\alpha m_\alpha d_\alpha)(a_\beta m_\beta d_\beta)}^{a_i m_i d_i} = C_{(a_\alpha m_\alpha)(a_\beta m_\beta)}^{a_i m_i} B_{(a_\alpha d_\alpha)(a_\beta d_\beta)}^{a_i d_i}, \quad (2)$$

where $C_{(a_\alpha m_\alpha)(a_\beta m_\beta)}^{a_i m_i}$ denotes the Clebsch-Gordon coefficients (that is associated with the change of basis of the direct product of two vector spaces into a direct sum), defined via :

$$|a_\beta, m_\beta\rangle = \sum_{a_i, m_i, a_\alpha, m_\alpha} C_{(a_\alpha m_\alpha)(a_\beta m_\beta)}^{a_i m_i} |a_i, m_i\rangle |a_\alpha, m_\alpha\rangle. \quad (3)$$

The entries of $B_{(a_\alpha d_\alpha)(a_\beta d_\beta)}^{a_i d_i}$, the junk part of the MPS matrices, are not determined by on-site symmetry G . We note that given a group G , we can always find at least one Schur covering group $\tilde{G} \supset G$ for which the linear and all projective representations of G lift to linear representations.

Moreover, Clebsch-Gordon coefficients beyond the usual $SU(2)$ can be obtained from existing techniques, such as in Ref. [13]. Therefore, our formalism can be easily and systematically applied for any given finite group G .

A-four invariant MPS for three-level physical spins. Let us now apply the construction to the non-Abelian group \mathbb{A}_4 , the order 12 group of even permutations of 4 elements and the ‘proper’ isometries of a tetrahedron which contains 12 rotations. This group has 3 one-dimensional linear irreducible representations- $\alpha_0, \alpha_1, \alpha_2$ and 1 three-dimensional linear irreducible representations- Γ . It also has 3 two-dimensional projective representations $(\Gamma'_0, \Gamma'_1, \Gamma'_2)$ [14]. The covering group which contains the linear and projective representations is the order 24 binary tetrahedral group \mathbb{T}' which is isomorphic to $SL(2, 3)$, the special linear group of two dimensional matrices over a field of three elements. The structure of the matrices in MPS for the wavefunction in the protected \mathbb{A}_4 phase is

$$\begin{pmatrix} A^x = \sigma_x \otimes B \\ A^y = \sigma_y \otimes VBV^\dagger \\ A^z = \sigma_z \otimes V^\dagger BV \end{pmatrix}; B = \begin{pmatrix} B_{00} & B_{01} & B_{02} \\ B_{10} & B_{11} & B_{12} \\ B_{20} & B_{21} & B_{22} \end{pmatrix}; V = \begin{pmatrix} \mathbb{1}_0 & 0 & 0 \\ 0 & \omega \mathbb{1}_1 & 0 \\ 0 & 0 & \omega^* \mathbb{1}_2 \end{pmatrix}, \quad (4)$$

and $\mathbb{1}_{i=0,1,2}$ is a $\dim(\mathbb{D}_i) \times \dim(\mathbb{D}_i)$ identity matrix, and the size of the junk matrices B_{ij} is $\dim(\mathbb{D}_i) \times \dim(\mathbb{D}_j)$, the degeneracies of vector space of the representation $\Gamma'_{i=0,1,2}$. We see that each MPS matrix splits up into a protected Pauli matrix and a junk matrix. We cannot further constrain the junk part, even with inversion nor with time-reversal symmetry. We remark that considering a larger group S_4 , Miller and Miyake were able to use further conditional measurement to probabilistically construct universal one-qubit gates [15].

A model two-body Hamiltonian with A-four and inversion symmetry. A natural question arises: whether there exist a Hamiltonian to realize such non-trivial SPT phase? Here, we construct a Hamiltonian to realize this phase. The AKLT state is already the unique ground state of a particular combination of the invariant form, and thus there are two other types of interaction in the total Hamiltonian

$$H_{\text{total}} = H_{AKLT} + \lambda H_c + \mu H_q, \quad H_{AKLT} = \sum_i \left[\vec{S}_i \cdot \vec{S}_{i+1} + \frac{1}{3} (\vec{S}_i \cdot \vec{S}_{i+1})^2 \right],$$

$$H_q = \sum_i \left[(\vec{S}_i^2 \cdot \vec{S}_{i+1}^2) - \frac{1}{3} (\vec{S}_i \cdot \vec{S}_{i+1})^2 \right], \quad H_c = \sum_{i;(\alpha,\beta,\gamma)=\Pi(x,y,z)} \left[(S^\alpha S^\beta)_i S_{i+1}^\gamma + S_i^\alpha (S^\beta S^\gamma)_{i+1} \right].$$

where i labels sites and Π denotes permutation (of three objects) and $\vec{S}_i^2 \cdot \vec{S}_{i+1}^2 \equiv (S_i^x S_{i+1}^x)^2 + (S_i^y S_{i+1}^y)^2 + (S_i^z S_{i+1}^z)^2$. Shown in Fig. 1 there is an extended region in this Hamiltonian such that the ground state is uniquely the AKLT state. Outside this region \mathbb{A}_4 is broken down to $\mathbb{Z}_2 \otimes \mathbb{Z}_2$, where the identity gate is still protected. For this particular Hamiltonian, the entire \mathbb{A}_4 -inversion symmetric region incidentally possesses the AKLT state as the ground state. However, we note that this is not generic.

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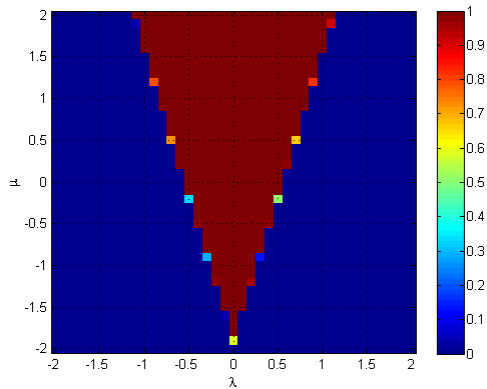


FIG. 1. Fidelity of ground states with the AKLT state. In the wedge region where the color is red i.e. the fidelity is equal to unity, the ground state is uniquely the AKLT state. Accidentally, for the Hamiltonian considered here, the ground state in the entire \mathbb{A}_4 +inversion symmetric phase is AKLT state, even though it is not generic.

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