Network coding for distributed quantum computation over the butterfly and cluster networks

Seiseki Akibue* and Mio Murao*,†

*Department of Physics, Graduate School of Science, The University of Tokyo
†NanoQuine, The University of Tokyo, Tokyo, Japan

Abstract

To apply *network coding for quantum computation*, we study distributed implementation of unitary operations over all separated input and output nodes of quantum networks. We consider a setting of networks where quantum communication between nodes are restricted to sending just a single qubit, but classical communication are freely allowed. We analyze which class of N-qubit unitary operations is implementable over *cluster networks* by investigating transformations of a cluster network into quantum circuits. We show that any two-qubit unitary operation is implementable over the *butterfly network* and the *grail network*, which are fundamental primitive networks for classical network coding. We also analyze probabilistic implementations of unitary operations over cluster networks.

1 Network coding for computation

Distributed quantum computation over multiple spatially separated quantum systems represented by nodes connected by mediating quantum systems represented by edges, is one of the most promising candidates for scalable quantum computation. A serious problem for any kind of distributed computation is the *bottleneck* problem, which is caused by the collision of several communication pathways between the nodes. The bottleneck problem worsens as the scale and the complexity of a communication network grow. Thus it is important to consider how to optimize transmission protocols so that the amount of quantum communications is reduced. In classical network information theory, *network coding* that incorporates processing at each network node in addition to routing provides efficient transmission protocols that can resolve the bottleneck problem [1].

Quantum communication with *quantum network coding* has been studied in analogy to classical network coding [2, 3, 4, 5]. k-pair quantum communication over a network is a unicast communication task to faithfully transmit a k-qubit state given at distinct input nodes $\{i_1, i_2, \cdots, i_k\}$ to distinct output nodes $\{o_1, o_2, \cdots, o_k\}$ through a given network. Two examples of 2-pair quantum communication over the *butterfly network* and the *grail network* are shown in Fig. 1. In k-pair quantum communication, the output state $|\text{output}\rangle_{o_1\cdots o_k}$ at the output nodes can be regarded as a state obtained by performing a k-qubit unitary operation U on the input state $|\text{input}\rangle_{i_1\cdots i_k}$ given at the input nodes

$$|\text{output}\rangle_{o_1\cdots o_k} = U\,|\text{input}\rangle_{i_1\cdots i_k}\,, \tag{1}$$

where U is a permutation operation. Without using network coding, both the butterfly network and the grail network exhibit the bottleneck problem, since one edge must be used twice to achieve 2-pair quantum communication. In the setting where classical communication is freely allowed between any nodes, however, it has been shown that there exists a quantum network coding protocol to achieve 2-pair quantum communication deterministically [3, 4, 5].

We do not need to restrict the k-qubit unitary operation U in Eq.(1) to be a permutation operation, but a general quantum operation. This leads to the idea of *network coding for quantum computation*, which aims to perform a quantum operation on a state given at distinct input nodes and to faithfully transmit the resulting state to the distinct output nodes efficiently over the network at the same time

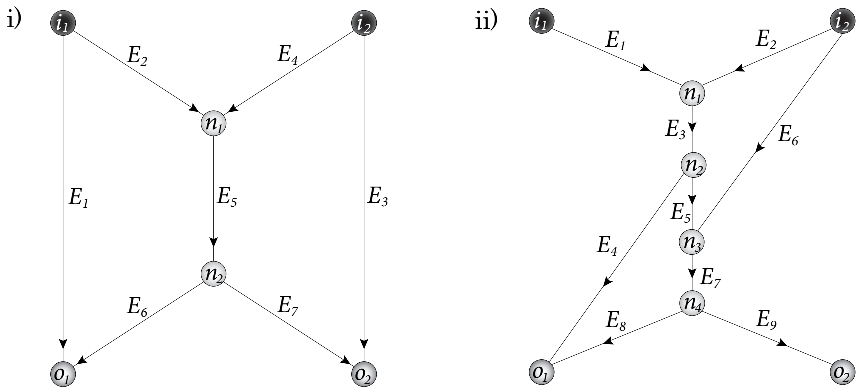

Figure 1: i) The butterfly network and ii) the grail network with the input nodes (i_1 and i_2), output nodes (o_1 and o_2) and the repeater nodes (n_1, n_2, n_3 and n_4). The directed edges $E_1, E_2, \cdots, E_9$ represent a single-qubit quantum channel. Our task is to transmit a given two-qubit state $|\mathrm{input}\rangle_{i_1,i_2}$ from i_1 to o_2 and from i_2 to o_1 simultaneously by using a single-qubit quantum channels and local quantum operations at each nodes.

[6, 7]. By computing and communicating simultaneously, quantum computation over the network may reduce communication resources in the distributed quantum computation scenario. As a first step to apply network coding for quantum computation, we investigate a *cluster network*, which is a special class of networks that have k input and k output nodes and an example of cluster networks is shown in Fig. 2 i). The cluster network contains the grail network as its special case. We concentrate on the setting that classical communication is freely allowed between any two nodes. This setting is justified in practical situations, where classical communication is much easier to implement experimentally than quantum communication. In this setting, a protocol for quantum network coding is equivalent to *local operations* (at each nodes) *and classical communication* (LOCC) or stochastic LOCC (SLOCC) assisted by entanglement given by the Bell pairs shared between nodes connected by edges, which is shown in Fig. 2 ii). We investigate what class of unitary operations is implementable by LOCC or SLOCC assisted by the entangled state corresponding to a given cluster network.

2 Results

First, we derive a method to convert a given network into quantum circuits, which are implementable over the given network by entanglement-assisted LOCC. Examples are given in Fig. 2 and Fig. 3. By analyzing the quantum circuit presented in Fig. 3, we show that any two-qubit unitary operation is deterministically implementable over both the butterfly network and the grail network. In previous works, only two-qubit unitary operations between different nodes had been used in quantum network coding protocols, however, we show that three-qubit unitary operations among three nodes are necessary for network coding to implement two-qubit unitary operations over the butterfly network. Next, we show that the set of all the implementable unitary operations over $(2, N)$-cluster and $(3, N)$-cluster network is obtained by using the conversion method. Third, we have analyzed probabilistic implementations of unitary operations over the cluster network.

3 Concluding remarks

A classical communication task over a general two input-output network is achievable if and only if the network contains one of primitive subgraphs, which are the butterfly subgraph, the grail subgraph and trivial paths [10]. The existence of such fundamental primitive subgraphs for quantum computational tasks has been an open problem. In this work, as a first step to find fundamental primitive subgraphs, we have shown that both the butterfly and grail networks are sufficient resources for implementing arbitrary two-qubit unitary operations, meanwhile a $(2, 2)$-cluster network is not sufficient.

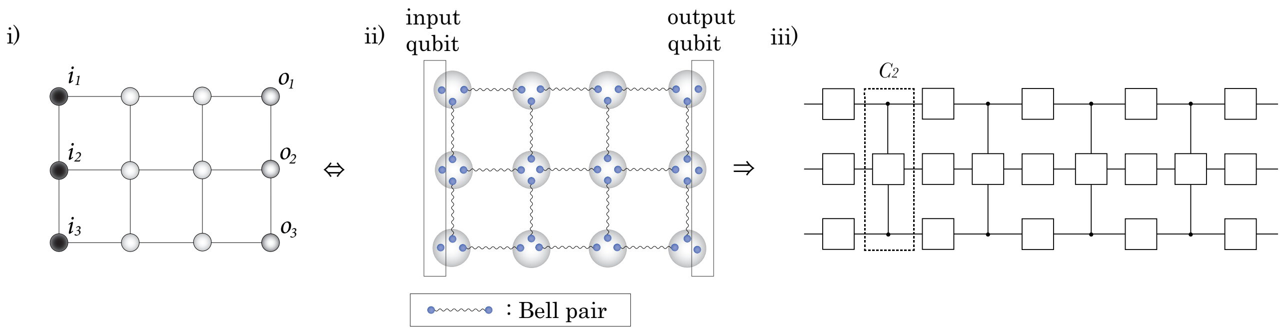

Figure 2: i) The $(3, 4)$-cluster network with the input nodes (i_1, i_2, i_3), output nodes (o_1, o_2, o_3) and 6 repeater nodes. The (k, N)-cluster network consists of k input and output nodes connected by horizontal wires, and N vertical wires. ii) The corresponding Bell pairs. A protocol for quantum network coding is equivalent to setting the state of input qubits, initializing the state of output qubits to $|0\rangle$, applying (S)LOCC to input qubits, Bell pairs and output qubits, and discarding all the qubits except output qubits. The grail network is equivalent to a $(2, 3)$-cluster network. Note that the entanglement resource for cluster networks is different from the cluster states used for measurement based quantum computation. iii) A converted quantum circuit consists of parallel three-qubit fully controlled unitary operations defined by $C_2 = |00\rangle\langle 00| \otimes u_{00} + |01\rangle\langle 01| \otimes u_{01} + |10\rangle\langle 10| \otimes u_{10} + |11\rangle\langle 11| \otimes u_{11}$, where $u_{ij} \in SU(2)$, and single qubit gates represented by small boxes.

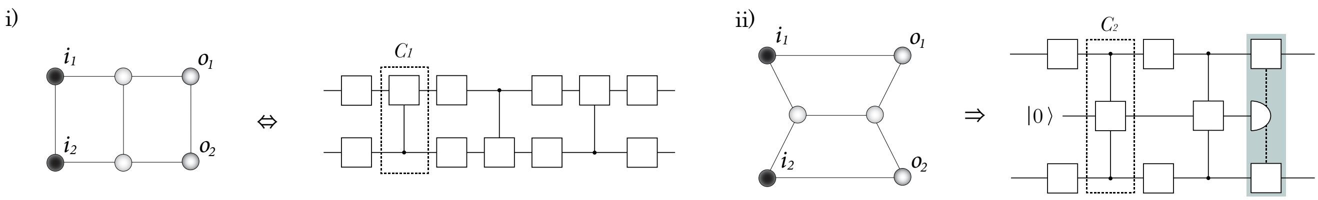

Figure 3: i) The $(2, 3)$-cluster network, which is equivalent to the grail network, and a converted circuit, where C_1 represents a two-qubit controlled unitary operation. Any two-qubit unitary operation is deterministically implementable on this converted circuit since the converted circuit can contain three CNOT gates and three CNOT gates are sufficient to implement arbitrary two-qubit unitary operations [11]. ii) We can apply our conversion method to the butterfly network with an ancillary qubit initialized to $|0\rangle$, a measurement and corrections corresponding to the shaded region of the quantum circuit, where C_2 represents a three-qubit fully controlled unitary operation defined in the figure caption of Fig. 2.

4 Acknowledgments

This work is supported by Project for Developing Innovation Systems of MEXT, Japan and JSPS by KAKENHI (Grant No. 23540463).

References

[1] R. Ahlswede, *et al.*, Network Information Flow 46, 1204 (2000).

[2] M. Hayashi, Phys. Rev. A 76, 040301(R) (2007).

[3] H. Kobayashi, *et al.*, LNCS 5555, 622 (2009).

[4] H. Kobayashi, *et al.*, ISIT, IEEE International Symposium on (2010).

[5] H. Kobayashi, *et al.*, ISIT, IEEE International Symposium on (2011).

[6] R. Appuswamy, *et al.*, IEEE Trans. Inf. Theory, vol.57, no.2, pp.1015-1030, (2011).

[7] A. Soeda, *et al.*, Phys. Rev. A 84, 012333 (2011).

[8] W. Dür, *et al.*, Phys. Rev. Lett. **89** 057901 (2002).

[9] B. Kraus, *et al.*, Phys. Rev. A 63, 062309 (2001).

[10] C.C. Wang, *et al.*, IEEE Trans. Inf. Theory, vol.56, pp. 3879-3900, (2010).

[11] Farrokh Vatan, *et al.*, Phys. Rev. A 69, 032315 (2004).