Robustness of spatial search by quantum walk

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1   Introduction

Quantum algorithms are prone to errors during their execution that may alter the outcome considerably. Although several error correcting schemes have been proposed, they require a considerable overhead in terms of resources. At this infant stage of quantum computing when only a limited number of qubits can be coherently controlled, a relevant question to be considered is: *How robust is a quantum algorithm against disorders when error correction is absent?*

In this talk, we address this question for the spatial search algorithm formulated as a continuous time quantum walk (CTQW) and show that it is robust to naturally occurring static errors. This algorithm was first developed by Childs *et al.* [1], where the authors show that searching a marked node $|w\rangle$, in a highly symmetrical graph such as a complete graph, can be done in $\mathcal{O}(\sqrt{N})$ time. The characteristics that a graph must possess for this algorithm to run optimally, remains an open question. In fact, in [2], where the authors present a different spatial search quantum algorithm based on the *divide and conquer* approach, their main criticism towards the CTQW version of the spatial search was the fact that an upper bound on the running time is not known, even if "minor defects are introduced". Furthermore, all graphs where this algorithm is known to hold optimally are regular, i.e., each node of such graphs is connected to an equal number of other nodes [1, 3].

The main contribution of this work is to show that the spatial search algorithm runs optimally not only when imperfections are introduced but also in non-regular graphs. First, we analytically show that, in the complete graph, this algorithm is robust to perturbations in the couplings between nodes. These perturbations can be large enough to amount to some links being broken completely (coupling goes to 0), thereby rendering the graph non-regular. Furthermore, we demonstrate that the algorithm optimally runs on complete bipartite graphs (CBG), which are in general, non regular. A particular case of the CBG, is the star graph where N-1 nodes are connected only to a central node. This is a planar graph, where N-1 nodes have connectivity 1, so it is surprising that the search works for such low connectivity structures.

At the heart of our analysis is a systematic method to reduce the dimension of the Hamiltonian of the problem, exploiting its invariant subspaces, which is sufficient to completely capture the relevant dynamics. This simplifies our analysis considerably for the graphs mentioned above. The method consists of projecting the Hamiltonian to a subspace of the full Hilbert space which is spanned by the set of states that will couple to the marked state $|w\rangle$ throughout evolution. This subspace is defined as the subspace containing $|w\rangle$ that is invariant under the action of the system Hamiltonian, and is known as a Krylov subspace. This reduction method is general and can be used to simplify the analysis of the quantum dynamics in systems that possess some symmetry. For example, we prove that this subspace is equal to the subspace spanned by the eigenstates of the Hamiltonian which have a non-zero overlap with $|w\rangle$, referred to as the 'non-invariant subspace' in [4] wherein $|w\rangle$ is the trapping site where an exciton gets absorbed in the quantum transport scenario. The calculation of this subspace is important for computing the transport efficiency in various networks, and for determining whether disorder and decoherence can aid transport [4, 5]. Our approach provides a simpler method to calculate this subspace which eliminates the need to compute the eigenstates of the full Hamiltonian. This way, the method is useful also for quantum transport problems.

Finally, we analyse the performance of the CTQW search algorithm in the presence of disorder in all the couplings of the complete graph and the star graph. Here, the reduction method does not work because there is no remaining symmetry and thus we have to rely on numerical results. For a complete graph with 1000 nodes, disorder in the couplings of up to 10% does not visibly affect the running time (increase of $\mathcal{O}(10^{-5})$). On the other hand, for the star graph with the same number of nodes, the same amount of disorder causes an increase in the running time of about 10%. It is remarkable that the algorithm tolerates such considerable imperfections without severely affecting its running time. We support the robustness observed numerically with theoretical insights on why the algorithm is robust in both graphs and why the complete graph is more robust than the star graph.

2 Main results

Hamiltonian reduction: Let $|w\rangle \in \mathcal{H}$ be a quantum state and $H \in \mathcal{L}(\mathcal{H})$ be the system's Hamiltonian, of dimension N. The unitary evolution of the state $|w\rangle$ can be expressed as

$$U(t)|w\rangle = \exp(-iHt)|w\rangle = \sum_{k=0}^{\infty} \frac{(-it)^k}{k!} H^k |w\rangle. \tag{1}$$

In many cases, if the Hamiltonian is highly symmetrical, only a small numbers of powers of $H^k |w\rangle$ are linearly independent. In general, the idea is to calculate the basis states that span the subspace generated by $H^k |w\rangle, \forall k > 0$. This subspace is the largest possible Krylov subspace [6], denoted by $\mathcal{K}_m(H, |w\rangle)$, where m is the dimension of the subspace which, for Hamiltonians with high symmetry, can be considerably smaller than N. Thus, $\mathcal{K}_m(H, |w\rangle)$ can be defined as the subspace of $\mathcal{H}$ which contains $|w\rangle$ and is invariant under the action of H. The projection of the Hamiltonian into this subspace yields a reduced Hamiltonian which captures the relevant dynamics.

k-link perturbation: When one link of the Hamiltonian of a complete graph of N nodes is perturbed, we obtain a three dimensional effective Hamiltonian after using the reduction process mentioned above. In the basis $\{|w\rangle, |s_{n-3}\rangle, |s_{ij}\rangle\}$ where $|w\rangle$ is the solution state, $|s_{ij}\rangle$ is the equal superposition of the nodes that have a perturbed coupling v, and $|s_{n-3}\rangle = \frac{1}{\sqrt{N-3}} \sum_{q \neq i,j,w} |q\rangle$ is the equal superposition of the rest of the nodes, the resulting Hamiltonian is

$$H_{red} = \begin{pmatrix} 0 & \sqrt{N-3} & \sqrt{2} \\ \sqrt{N-3} & N-4 & \sqrt{2(N-3)} \\ \sqrt{2} & \sqrt{2(N-3)} & v \end{pmatrix}. \tag{2}$$

In the limit of large N, we can use degenerate perturbation theory to show that the search Hamiltonian $H_{search} = -\gamma H_{red} - |w\rangle\langle w|$ runs optimally in spite of this perturbation. The critical value of $\gamma = 1/N$ renders $\langle w|H|w\rangle$ and $\langle s_{n-3}|H|s_{n-3}\rangle$ degenerate having values -1. The initial state of the algorithm is $|\psi_0\rangle \approx |s_{n-3}\rangle$, for large N and H_{search} can be expressed as

$$H_{search} = H + H_{pert} = \begin{pmatrix} -1 & 1/\sqrt{N} & 0 \\ 1/\sqrt{N} & -1 & \sqrt{2/N} \\ 0 & \sqrt{2/N} & 0 \end{pmatrix} + O(\frac{1}{N}). \tag{3}$$

The running time depends on the difference of the two lowest eigenvalues which are $E_{\pm} = -1 \pm 1/\sqrt{N}$ and is $T = \frac{\pi\sqrt{N}}{2}$.

Another example that is analytically treatable with this approach, is when k links are removed from the graph such that, for each node, at most one link connecting to this node is removed. Let Ω_k be the set of all nodes from which a link was broken. For simplicity, we do not consider the possibility of breaking a link connected to $|w\rangle$. The reduction procedure yields that the dynamics is restricted to the states $|w\rangle$, the equal superposition of all nodes that correspond to a broken link $|s_k\rangle = \frac{1}{\sqrt{2k}} \sum_{i_j \in \Omega_k} |i_j\rangle$ and the equal superposition of all other nodes $|s_{\bar{k}}\rangle = \frac{1}{\sqrt{N-2k-1}} \sum_{l \notin \Omega_k \cup \{w\}} |l\rangle$. In the basis $\{|w\rangle, |s_{\bar{k}}\rangle, |s_k\rangle\}$ the reduced Hamiltonian can be written as:

$$H_{red} = \begin{pmatrix} 0 & \sqrt{N-2k-1} & \sqrt{2k} \\ \sqrt{N-2k-1} & N-2k-2 & \sqrt{2k(N-2k-1)} \\ \sqrt{2k} & \sqrt{2k(N-2k-1)} & 2k-2 \end{pmatrix} \tag{4}$$
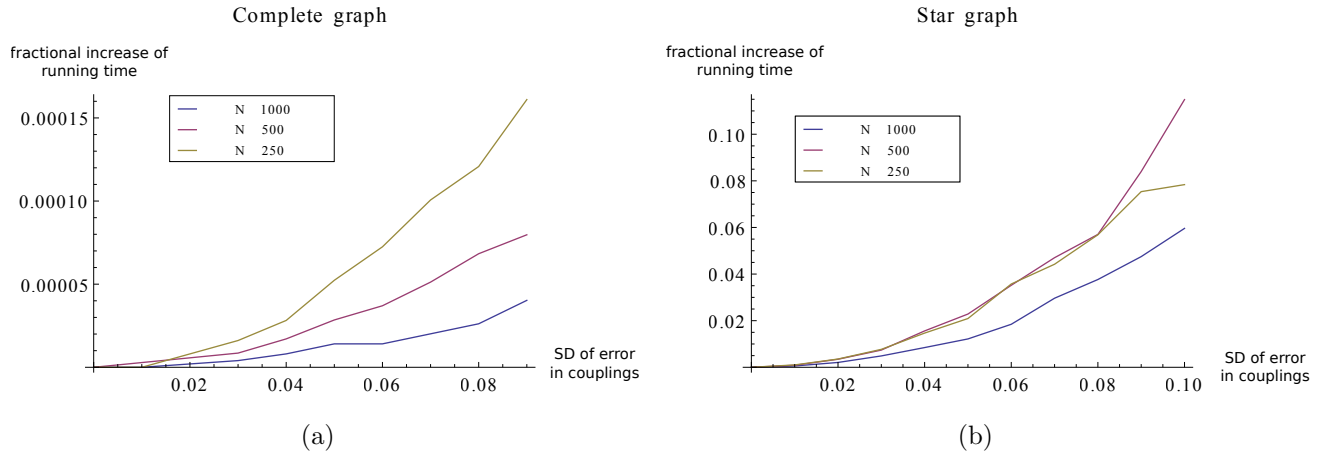
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Figure 1: Plot of fractional increase in running time versus the standard deviation of errors in the couplings complete graph and the star graph.

Since none of these states can, in general, be approximated to the initial superposition of states $|s\rangle$, we do a unitary transformation, similar to [3], to the basis $B = \{|w\rangle, |r\rangle, |r^\perp\rangle\}$. Here, $|r\rangle$ is defined as $|r\rangle = \frac{1}{\sqrt{N-1}} \sum_{q=1; q \neq w}^{N} |q\rangle$ i.e., the equal superposition of all nodes of the graph except the solution and $|r^\perp\rangle$ is a state orthogonal to $|r\rangle$ and $|w\rangle$. Using the same critical value for $\gamma = 1/N$, the search Hamiltonian is similar the complete graph and the effects of broken links is shown to be an effect of $\mathcal{O}(1/N)$. Thus, we obtain the same running time within this approximation.

Spatial search on complete bipartite graphs: A complete bipartite graph $G(V_1, V_2, E)$ has two sets of vertices V_1 and V_2 such that each vertex of V_1 is not connected to any other vertex in V_1 and is connected to all vertices of V_2 and vice-versa. This set of graphs is also denoted as $\mathcal{K}_{m_1, m_2}$, where $m_1 = |V_1|$ and $m_2 = |V_2|$ and we have $m_1 + m_2 = N$. This is a non-regular graph, as long as $m_1 \neq m_2$. We show that the dynamics can be restricted to a 3 dimensional subspace and, defining $m_1 = \alpha N$ and $m_2 = (1 - \alpha)N$, we obtain the expected running time $T = \pi \sqrt{N}(1 + 2\sqrt{\alpha(1 - \alpha)})^{1/2}$. In the particular case where $m_1 = N - 1$ and $m_2 = 1$ we have a star graph which is a planar structure. One of the $N - 1$ nodes of degree 1 holds the solution. It is surprising that even in such a simple structure, search is optimal. Furthermore, we show analytically that search is robust up to order $\mathcal{O}(k/N)$ where k is the number of affected links, assuming that the link containing the solution node is not broken. This happens with probability $\mathcal{O}(1/N)$, which is low for large N.

Numerical results on robustness of complete graph and star graph: We numerically study the effect of introducing errors in all the couplings of a complete graph and a star graph, sampled from independent Gaussian distributions with mean 0 and standard deviation σ. In Fig. 1a, the increase in running time for the complete graph is plotted against the standard deviation in the disorder of the couplings. We observe that this structure is robust and this robustness, remarkably increases on increasing the number of nodes. In the case of a star graph, the robustness is not as strong as that of the complete graph as shown in Fig. 1b. However, as in the complete graph, it gets more robust on increasing the number of nodes. The robustness is considerable (the running time increases by only 10%) given its low connectivity and simplicity. Theoretically, the effect of the perturbation on the running time is proven to be $\mathcal{O}(\sigma^2/N)$ for a complete graph and $\mathcal{O}(\sigma^2/\sqrt{N})$ for a star graph, thereby justifying the observed results.

3   Conclusions

We could analytically and numerically show that a continuous time quantum algorithm is robust against considerable perturbations. This is the first example where an algorithm has been shown to be robust in this paradigm. Furthermore, we reveal that the spatial search algorithm can optimally run even on non-regular graphs. This paves the way for future analysis of the robustness of other continuous time algorithms where error correction is absent. The method to systematically reduce the Hamiltonian dimensions can have varied applications ranging from analysis of continuous time quantum algorithms to quantum transport. This work will appear on arXiv soon.

Acknowledgements

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