Witnessing entanglement by proxy

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While the occurrence and possible uses of entanglement were first studied for bipartite states, entanglement in systems containing a large number of particles is of interest both from a theoretical and from a practical point of view. Even though the macroscopic world we experience daily, can be described classically, there are a number of systems that are large enough to be described by the thermodynamical limit, which exhibit quantum behaviour, Bose-Einstein condensates, ferromagnetic and superconducting materials being prominent examples. Entanglement may turn out useful in understanding thermodynamical phenomena such as phase transitions in such systems [1, 2]. Recently there has also been a lot of attention on the role of entanglement in cooling processes [3]. Other possible applications of large entangled systems are quantum computers based on solid state or NMR systems [4–7]. In addition to studying entanglement in the limit of many particles it is also worth asking up to which temperature entanglement can exist. This is an important question for experiments, where cooling down systems requires lots of resources. While entanglement usually exists at very small temperatures, it could persist to up to 100K in superconductors [8].
Experimentally detecting entanglement in macroscopic systems is generally a highly nontrivial task, even for NPT entanglement. Checking the PPT criterion, as easy as it is theoretically, requires a full state tomography, which is not possible in large systems. Also, calculating the eigenvalues for matrices of large dimensions is not practical. The method of choice are entanglement witnesses, i.e. observables with positive expectation value for all separable states but with negative expectation value for some entangled states. Witnesses reduce the complexity of entanglement detection to the measurement of a single observable. However, this observable might have no physical meaning and might be hard or impossible to measure. In particular it might be necessary to perform a collective measurement of all particles, which is not experimentally feasible in macroscopic systems. What is feasible is the measurement of macroscopic observables such as the internal energy, the temperature or the entropy of the system. There have been several results showing that internal energy and temperature can serve as entanglement witnesses at low temperatures ([9–14] to name just a few). We will now present an experimentally feasible method of entanglement detection that, by construction, works at higher internal energies than [9–11, 14]. Our method uses PPT, arbitrary entanglement witnesses or positive maps without the need to measure them directly. Instead, it is sufficient to measure the systems entropy and internal energy or temperature. Since those quantities do not witness the entanglement directly, we call them *proxy witnesses*.
Let us now briefly explain our method and state some results. For details please refer to the technical version in the appendix. The authors of [9–11] make use of the fact that any state with (internal) energy less than the minimum energy allowing for separable states has to be entangled. By convexity the minimum is attained at a pure state. Our idea is to add a lower bound on the (von Neumann) entropy as an additional constraint, i.e. we use that any state below

$$\min_{\rho\in\text{sep},\ S(\rho)\geq S_0} \operatorname{Tr}\rho H \tag{1}$$

has to be entangled. S_0 can be varied between 0 and $\log d$. It can be easily shown that we can detect the same states by computing

$$\max_{\rho\in\text{sep},\ \operatorname{Tr} H\rho\leq E} S(\rho) \tag{2}$$

and varying E. Any state with larger entropy has to be entangled. Note that (2) is just the optimisation yielding the Gibbs state with the additional constraint of separability. In particular, if the Gibbs state entropy is larger than (2), we know that the Gibbs state is entangled.

The optimisation in (2) however is difficult to deal with because of the separability constraint. Our main idea is to relax this constraint using sets of states which remain positive semidefinite after the application of positive (yet not completely positive) maps Λ. While these are of course supersets of the separable states, it is clear that for every entangled state in principle there exists a map Λ and thus a semidefinite relaxation that will still yield optimal results for the constrained optimisation.

$$\begin{aligned} \max S(\rho) \quad \text{s.t.}\ \rho \geq 0,\ \operatorname{Tr}\rho = 1,\ \operatorname{Tr}\rho H \leq E, \\ \Lambda_A \otimes \mathrm{id}_B(\rho) \geq 0, \end{aligned} \tag{3}$$

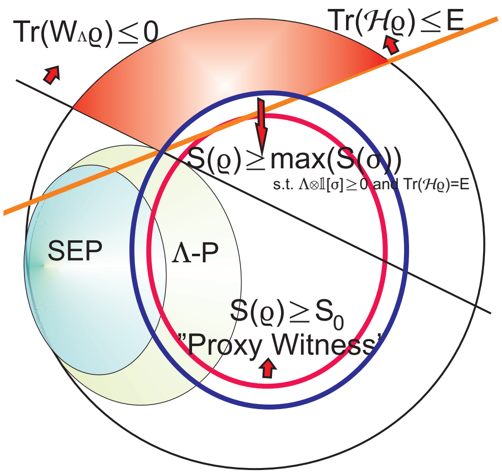


FIG. 1. Here the principal idea of the constrained maximum entropy principle as a proxy witness is illustrated. The set of separable states is approximated by states which are positive under application to a positive map to a subsystem (Λ-P in the figure) or by a suitable entanglement witness W_Λ. The macroscopically accessible information given by the line corresponding to the average energy (orange) and the entropy (inner circle, red) can be used as a proxy for the witness that is inaccessible. This can be seen as the optimization over the Λ-P constrained set yields a smaller value (outer circle, blue) and the resulting entropy gap proves the presence of entanglement in the system.

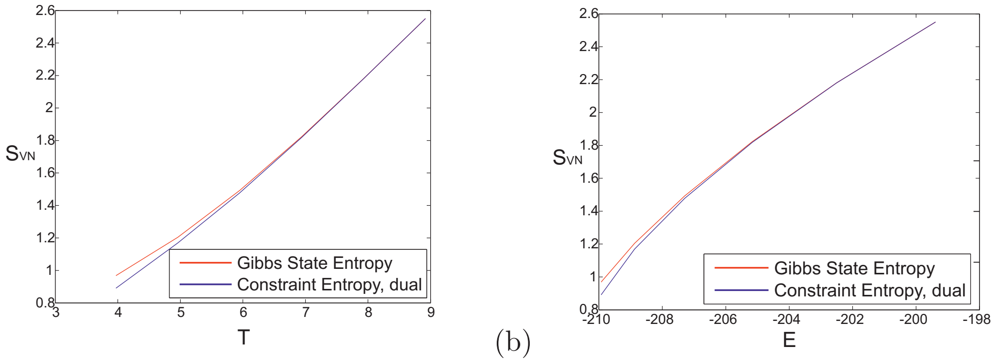
The most prominent example of a positive but not completely positive map is transposition, giving rise to the PPT criterion. Maps can also be obtained from any entanglement witness and vice versa. For an entanglement witness W, the positivity condition becomes $\operatorname{Tr}\rho W \geq 0$. Note that even though our criterion might make use of PPT or witnesses, neither a state tomography nor measurement of the witnesses is necessary to apply it. All that needs to be done is to perform the optimisation in (3) and measure the entropy and the energy of the system in question. Let us now have a look at the optimisation problem. It is a convex problem with linear and semidefinite constraints, hence can be solved efficiently with interior-point algorithms [15]. Piecewise linear approximation of the objective function combined with Ky Fan's principle even allows us to compute (3) with standard SDP solvers [16]. Moreover, such optimisation problems have a duality theory that can provide us with certified upper bounds on the maximum in (3) [15]. Namely, numerical calculation of the dual of (3), even if only performed approximately, yields checkable dual feasible points which give us an analytical upper bound on (3). The dual problem is equivalent to solving

$$\min \ell(\mu, X_{A,\Lambda}) \quad \text{s.t.}\ \mu \in \mathbb{R},\ X_{A,\Lambda} \geq 0, \tag{4}$$

where

$$\ell(\mu, X_{A,\Lambda}) = \log \operatorname{Tr} \exp \left(\mu H + \sum_A \Lambda^* \otimes \mathrm{id}(X_{A,\Lambda}) \right) - \mu E. \tag{5}$$

Using the corresponding witnesses W_i, the problem reduces to minimising $\log \operatorname{Tr} \exp \left(\mu H + \sum_i \operatorname{Tr} \nu_i W_i \right) -$



(a) (b)

FIG. 2. As an exemplary application of our proxy witness method we plot the results for 13 qubits with an Heisenberg XXZ model interaction strength using a single witness and $J_x = 13$, $J_z = 1$ and $B = -1$. In (a)/(b) we plot the entropy/energy of thermal states and the dual of our constrained entropy proxy versus temperature. Where the Gibbs state entropy is larger than the dual of the constraint entropy we detect entanglement in regions where the energy alone would actually be compatible with the system being separable.

μE over all $\nu_i \geq 0$. The ν_i are scalar variables, greatly simplifying the optimisation.

Using a linear version of the witness given in [17], we are able to detect entanglement in a XXZ Heisenberg system in a magnetic field for up to 13 qubits (Depicted in figure 2). The choice of the witness, which is designed to detect entanglement in Dicke states [18], is motivated by the fact that the ground states of XXZ Heisenberg Hamiltonians with coupling constant $J_z > 0$ are Dicke states [19]. We can also detect entanglement of the antiferromagnetic Heisenberg model for up to five qubits using PPT.

Going to the thermodynamical limit, we can no longer make use of numerical optimisation of (5). In order to find the minimum in (4) analytically, let us focus on witnesses of the form $W = \alpha\mathbb{1} - |E_0\rangle\langle E_0|$, where $|E_0\rangle$ is the ground state of H and $\alpha = \max_{|\phi\rangle,|\varphi\rangle} |\langle E_0||\phi\rangle|\varphi\rangle|^2$. We can either maximise over a fixed bipartition, or over all possible ones. In the former case we can detect bipartite entanglement w.r.t. the partition chosen, in the latter case we can detect genuinely multipartite entanglement. Inserting the witness into (5), the exponent becomes diagonal. This allows us to show that for $0 < \alpha < \frac{e^{-\beta E_0}}{Z}$ there is a nonzero gap between the Gibbs state entropy and the dual of the constraint entropy. Hence not only the Gibbs state is shown to be entangled, which could also be shown by applying the witness directly, but also states with almost maximal entropy. Note that our criterion only depends on α, the ground state energy and the partition function, which are known for a number of systems in the thermodynamical limit [20].

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