

Rényi generalizations of quantum information measures

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Summary. Our main result is a procedure to obtain a Rényi generalization of any quantum information measure which consists of (a) a linear combination of von Neumann entropies with coefficients chosen from the set $\{-1, 0, 1\}$ [2], or, (b) a difference of two relative entropies [25]. In [2], we apply the procedure to obtain Rényi generalizations of the conditional quantum mutual information (CQMI). We can show that the proposed Rényi CQMIs retain the desired properties of the original quantity based on the von Neumann entropy. We conjecture that the proposed Rényi CQMIs are monotone increasing in the Rényi parameter and give proofs of the conjecture for some special cases. The solution of the conjecture would imply a characterization of quantum states with small CQMI. This characterization would be useful for understanding topological order in condensed matter physics [10, 11], for solving some open questions related to squashed entanglement [32], as well as for deriving quantum communication complexity lower bounds [12], as discussed in [30].

The CQMI underlies the squashed entanglement (SE) [3] and the quantum discord (QD) [20]. We define a Rényi SE and a Rényi QD [2, Section 10], and investigate some properties and applications of the quantities [25]. In particular, we investigate the validity of the proposed Rényi SE as an entanglement measure [9, 22] and we show that it retains many of the properties of the von Neumann quantity. The results in [2] and [25] involve the use of trace inequalities from matrix analysis including recently invented proof techniques of Hiai [7] and Frank and Lieb [4].

Background. In quantum information theory, there has been much interest lately towards generalizing quantum information measures in terms of the Rényi entropies [24]. Rényi generalizations of quantum information measures enable us to improve our understanding of the original von Neumann quantities; e.g., in some cases they serve as crucial tools in proving that the corresponding von Neumann quantities are strong converse rates for certain operational tasks [1, 18, 13, 23, 26, 31, 5, 29]. Rényi quantum information measures also find applications in scenarios beyond the traditional i.i.d. (independent and identically distributed) resource setting, e.g., in the one-shot regime [8, 6, 15, 16].

A quantum Rényi conditional entropy and mutual information have been proposed and studied [18, 13, 15, 26, 17, 31, 5, 28]. However, it has been a vexing open question to obtain a Rényi conditional quantum mutual information (CQMI). The CQMI of a tripartite state ρ_{ABC} is defined as

$$I(A; B|C)_\rho \equiv H(AC)_\rho + H(BC)_\rho - H(C)_\rho - H(ABC)_\rho, \quad (1)$$

where $H(F)_\sigma \equiv -\text{Tr}\{\sigma_F \log \sigma_F\}$ is the von Neumann entropy of a state σ_F on system F . The CQMI is non-negative and monotone under local quantum operations on systems A and B . The CQMI also obeys a duality relation, $I(A; B|C)_\psi = I(A; B|D)_\psi$, for any four-party pure state ψ_{ABCD} .

Validation criteria for Rényi generalizations. We consider the following criteria to validate Rényi generalizations of quantum information measures: (a) A Rényi generalization should retain the essential, desired properties of the original von Neumann quantity; e.g., in the case of the CQMI, these properties include non-negativity, monotonicity under local quantum operations on systems A and B , and the duality relation mentioned above. (b) A Rényi generalization should converge to the original von Neumann quantity in a suitable limit of the Rényi parameter. This standard already invalidates the most commonly employed Rényi generalizations of measures that are linear combinations of von Neumann

entropies—namely, where the von Neumann entropies are simply replaced by Rényi entropies. For example, the following Rényi CQMI is invalid according to the above standard:

$$I'_\alpha(A; B|C)_\rho \equiv H_\alpha(AC)_\rho + H_\alpha(BC)_\rho - H_\alpha(C)_\rho - H_\alpha(ABC)_\rho, \quad (2)$$

where $H_\alpha(F)_\sigma \equiv [1 - \alpha]^{-1} \log \text{Tr} \{ \sigma_F^\alpha \}$ for a state σ_F on system F , and $\alpha \in (0, 1) \cup (1, \infty)$. This is because, in general, $I'_\alpha(A; B|C)_\rho$ can be negative. Furthermore, the results of [14] imply that there are generally no linear inequality constraints on the marginal Rényi entropies of a multiparty quantum state other than non-negativity when $\alpha \in (0, 1) \cup (1, \infty)$. This also implies that monotonicity under local quantum operations generally does not hold for $I'_\alpha(A; B|C)_\rho$, and [14] provides many examples of four-party states ρ_{ABCD} such that $I'_\alpha(A; BD|C)_\rho < I'_\alpha(A; B|C)_\rho$.

Procedure. Our procedure to obtain a Rényi generalization of a linear combination of von Neumann entropies, detailed in [2, Section 10], can be summarized in three steps. (a) We write the quantum information measure that we wish to “Rényi generalize” as a single information quantity in terms of the relative entropy. (b) We re-express the second argument of the relative entropy using the generalized Lie-Trotter product formula [27]; this introduces an extra parameter α . (c) We use the free parameter α as our handle and replace the relative entropy with the Rényi relative entropy [21], [17], [31] or the sandwiched Rényi relative entropy [17], [31] by allowing α to vary within suitable intervals. The resulting quantity is a Rényi generalization of the information measure. The procedure to obtain a Rényi generalization of a quantity which is a difference of two relative entropies is very similar, and is given in [25]. In short, we re-express the relative entropy difference as a single relative entropy and apply steps (b) and (c) outlined above.

We now show how to apply the above procedure to obtain Rényi generalizations of the CQMI.

Definition 1. *The Rényi conditional mutual information of ρ_{ABC} is defined for $\alpha \in [0, 1) \cup (1, \infty)$ as*

$$I_\alpha(A; B|C)_\rho \equiv \inf_{\sigma_{BC}} \frac{1}{\alpha - 1} \log \text{Tr} \left\{ \rho_{ABC}^\alpha \rho_{AC}^{(1-\alpha)/2} \rho_C^{(\alpha-1)/2} \sigma_{BC}^{1-\alpha} \rho_C^{(\alpha-1)/2} \rho_{AC}^{(1-\alpha)/2} \right\}, \quad (3)$$

where the optimization is over density operators σ_{BC} such that $\text{supp}(\rho_{ABC}) \subseteq \text{supp}(\sigma_{BC})$ and the matrix inverses are understood to be generalized inverses.

We can identify an explicit form for the minimizing σ_{BC} and thus for $I_\alpha(A; B|C)_\rho$ when $\alpha \in (0, 1) \cup (1, \infty)$, and write the latter as

$$I_\alpha(A; B|C)_\rho = \frac{\alpha}{\alpha - 1} \log \text{Tr} \left\{ \left(\rho_C^{(\alpha-1)/2} \text{Tr}_A \left\{ \rho_{AC}^{(1-\alpha)/2} \rho_{ABC}^\alpha \rho_{AC}^{(1-\alpha)/2} \right\} \rho_C^{(\alpha-1)/2} \right)^{1/\alpha} \right\}. \quad (4)$$

This follows because the infimum in (3) can be replaced by a minimum and the minimum σ_{BC} is unique with an explicit form given by a Sibson identity [2]. The sandwiched Rényi CQMI is similarly defined in [2, Definition 19]. In [2], we have proven that these Rényi CQMIs satisfy most of the criteria for a valid Rényi generalization mentioned before (see Table 1 for a summary of the properties that hold and some which remain open, however with numerical evidence supporting them).

Applications of the proposed Rényi CQMI.

- **Monotonicity in the Rényi parameter.** We conjecture that *our Rényi generalizations of the CQMI are monotonically increasing functions of the Rényi parameter*. That is, for $0 \leq \alpha \leq \beta$, we conjecture that

$$I_\alpha(A; B|C)_\rho \leq I_\beta(A; B|C)_\rho, \quad (5)$$

as well as the analogous statement for the sandwiched Rényi CQMI. These conjectures are true in a number of special cases. For example, we show that these conjectures hold when the Rényi parameter

Formula	$I(A;B C)_\rho$ (1)	$I'_\alpha(A;B C)_\rho$ (2)	$I_\alpha(A;B C)_\rho$ (4)	$\tilde{I}_\alpha(A;B C)_\rho$ ([2, (6.13)])
Non-negative	✓	✗	✓	✓
Monotone under local op.'s on A	✓	✗	?	?
Monotone under local op.'s on B	✓	✗	✓	✓
Duality	✓	✓	✓	✓
Monotone in α	N/A	✗	?	?

Table 1: A comparison between the original von Neumann CQMI, the commonly considered Rényi generalization of the quantity, and the Rényi generalizations proposed in this work, in terms of some properties of the original CQMI. The ?'s indicate open questions with numerical evidence supporting a positive answer.

α is in a neighborhood of one, and that (5) is true in the case when $\alpha + \beta = 2$ (or when $1/\alpha + 1/\beta = 2$ for the sandwiched Rényi CQMI) [2, Section 8]. If proven to be correct generally, the above conjecture would establish the truth of an open conjecture from [10] (up to a constant):

$$I(A;B|C)_\rho \geq \tilde{I}_{1/2}(A;B|C)_\rho = -\log F(\rho_{ABC}, \mathcal{R}_{C \rightarrow AC}^P(\rho_{BC})) \quad (6)$$

$$\geq \frac{1}{4} \|\rho_{ABC} - \mathcal{R}_{C \rightarrow AC}^P(\rho_{BC})\|_1^2 \quad (7)$$

where $\mathcal{R}(\cdot) \equiv \rho_{AC}^{1/2} \rho_C^{-1/2} (\cdot) \rho_C^{-1/2} \rho_{AC}^{1/2}$ denotes Petz's recovery map for the partial trace over A [19] and $F(P, Q) \equiv \|\sqrt{P} \sqrt{Q}\|_1^2$ is the quantum fidelity. This would give an operational characterization of quantum states with small CQMI (i.e., states that fulfill strong subadditivity with near equality) [2, Section 8.4]. Some important applications of such a characterization were pointed out in the first paragraph of this submission.

- **Rényi squashed entanglement** We define the Rényi SE of a bipartite state ρ_{AB} as

$$E_\alpha^{\text{sq}}(A;B)_\rho \equiv \frac{1}{2} \inf_{\omega_{ABE}} \{I_\alpha(A;B|E)_\omega : \rho_{AB} = \text{Tr}_E \{\omega_{ABE}\}\}, \quad (8)$$

where the infimum is over all tripartite extensions ω_{ABE} of the state ρ_{AB} . We prove that the above functional is non-negative, monotone under LOCC operations, convex, additive on tensor-product states, and takes the value zero iff the state ρ_{AB} is a separable state. (Some of these proofs rely on the monotonicity of the Rényi CQMI under local operations on system A , which however is yet to be proven.) These properties qualify it as an entanglement measure in itself. We show that the Rényi SE is a lower bound on the Rényi entanglement of formation of bipartite states. The Rényi SE could potentially be used to prove strong converse bounds for entanglement distillation and two-way assisted quantum communication.

- **Rényi quantum discord** QD, which is traditionally viewed as the gap between total correlations and classical correlations in a bipartite state, can also be viewed as the following CQMI

$$D(A;B)_\rho = \inf_{\{\Lambda\}} I(B;E|X),$$

where the optimization is over all POVMs acting on the system A , with X being the classical output and E being an environment system that forms an isometric extension of any such measurement [2]. This allows us to define a Rényi QD by considering the Rényi CQMI of (3) in place of the von Neumann CQMI. In [25], we prove that a Rényi QD defined in such a way retains the desired properties of the von Neumann-based QD, namely, it is non-negative and invariant under local unitary operations.

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