Limitations on separable measurements by convex optimization

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The paradigm of *local operations and classical communication*, or *LOCC* for short, is fundamental within the theory of quantum information. A protocol involving two or more individuals is said to be an *LOCC protocol* when it may be implemented by means of classical communication among the individuals, along with arbitrary quantum operations performed locally. Within the LOCC paradigm, the problem of *discrimination of quantum states (bipartite or multipartite)* has been extensively studied. In the most typically considered variant of this problem, the two parties are given a single copy of a quantum bipartite state chosen with some probability from a known collection of states and their goal is to identify which state was given by means of an LOCC measurement.

Many examples are known of specific choices of pure, orthogonal states for which a perfect discrimination is not possible through LOCC measurements. Some of these examples, along with other general results concerning this problem, may be found in [BW09, BDF+99, DFXY09, Fan04, GKR+01, GKRS04, HMM+06, HSSH03, Nat05, WH02, WSHV00, Wat05, YDY12, YDY14].

As perhaps the simplest example of an instance of this problem where a perfect LOCC discrimination is not possible, one has that the four standard Bell states cannot be perfectly discriminated by LOCC measurements [GKR+01]. In particular, if the states are selected with uniform probability, it holds that the maximum probability of distinguishing them via LOCC is $1/2$. Among the other known examples of collections of orthogonal pure states that cannot be perfectly discriminated by LOCC protocols, the so-called *domino state* example of [BDF+99] is noteworthy. The particular relevance of this example lies in the fact that all of these states are product states, demonstrating that entanglement is not a requisite for a set of orthogonal pure states to fail to be perfectly discriminated by any LOCC measurement.

The set of measurements that can be implemented through LOCC has an apparently complex mathematical structure—no tractable characterization of this set is known, representing a clear obstacle to a better understanding of the limitations of LOCC measurements. For this reason, the state discrimination problem described above is sometimes considered for more tractable classes of measurements that approximate, in some sense, the LOCC measurements. The classes of *positive-partial-transpose* (PPT) and *separable* measurements represent two commonly studied

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approximations in this category. Our results are primarily focused on separable measurements. Given two complex Euclidean spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively denoting Alice's and Bob's subspaces, a positive semidefinite operator $P \in \text{Pos}(\mathcal{X} \otimes \mathcal{Y})$ is said to be *separable* if it is possible to write

$$P = \sum_{k=1}^{M} Q_k \otimes R_k \tag{1}$$

for some choice of a positive integer M and positive semidefinite operators $Q_1, \ldots, Q_M \in \text{Pos}(\mathcal{X})$ and $R_1, \ldots, R_M \in \text{Pos}(\mathcal{Y})$; and a measurement $\{P_1, \ldots, P_N\}$ on $\mathcal{X} \otimes \mathcal{Y}$ is said to be a separable measurement if it is the case that each measurement operator P_k is separable. It is straightforward to prove that every LOCC measurement is a separable measurement and every separable measurement is in turn a PPT measurement. It follows that any limitation proved to hold for every separable or PPT measurement also hold for every LOCC measurement.

The state discrimination problem for separable measurements has been investigated before [DFXY09, BN13], although many natural questions remain still open. In this work, we apply techniques from convex optimization in order to answer several of these questions. A nice mathematical property of the set of separable measurements is the fact that an optimization over this set can be cast in the framework of cone programming, which is a generalization of linear programming and semidefinite programming that allows for optimizations over general closed, convex cones. From a computational point of view, the characterization of the problem as a cone program does not seem to help, as optimizing over the set of separable measurements is known to be an NP-hard problem. However, many results may be derived analytically through the notion of duality of cone programs. By duality theory, one may obtain upper bounds on the success probability of any separable measurement for a given state discrimination problem. A central role in our results is played by the dual cone to the set of separable operators, which is the set of *block-positive operators*. These are exactly the linear operators that can be characterized as Choi representations of *positive* (not necessarily completely positive) linear maps. A major contribution of our work is that we establish a new connection between some families of well-known positive linear maps and some sets of locally indistinguishable states, which were independently studied.

The rest of the abstract is a summary of the applications of our techniques to three open questions concerning the discrimination of quantum states by separable measurements.

The entanglement cost of discriminating Bell states

We study the entanglement cost of discriminating Bell states. It is straightforward to see that two Bell states can be distinguished with perfect probability by LOCC protocols. However, we know that four Bell states can be discriminated with success probability at most $1/2$, while for three Bell states the success probability is at most $2/3$. These bounds hold not only for LOCC measurements, but also for separable and PPT measurements.

On the other hand, if Alice and Bob are provided with an ancillary maximally entangled pair of qubits, then they can apply a simple protocol based on teleportation to perfectly discriminate all four Bell states (and therefore the same is true for any three Bell states). It is known that a maximally entangled state of two qubits is also necessary to distinguish the four Bell states exactly. This leaves open the question of how much entanglement the parties need in the resource state to achieve perfect discrimination of sets of three Bell states. In this work, we obtain an exact formula for the optimal probability of correctly discriminating any set of either three or four Bell states via separable measurements, when the parties are given an ancillary partially entangled pair of qubits. In particular, it is proved that this ancillary pair of qubits must be maximally entangled in order for three Bell states to be perfectly discriminated by separable (or LOCC) measurements, which answers an open question of [YDY14]. In order to solve this problem, we introduce a new family of linear positive maps from $\mathrm{L}(\mathbb{C}^4)$ to $\mathrm{L}(\mathbb{C}^4)$, which may find applications elsewhere in quantum information theory.

State discrimination and unextendable product sets

We investigate the distinguishability of unextendable product sets. A set $\mathcal{A} \subset \mathcal{X} \otimes \mathcal{Y}$ of mutually orthogonal product states is said to be *unextendable* if it is impossible to find a nonzero product vector that is orthogonal to every element of $\mathcal{A}$. It is known that no unextendable product set $\mathcal{A} \subset \mathcal{X} \otimes \mathcal{Y}$ spanning a proper subspace of $\mathcal{X} \otimes \mathcal{Y}$ can be perfectly discriminated by an LOCC measurement [BDM+99], while every unextendable product set can be discriminated perfectly by a PPT measurement. It is also known that every unextendable product set $\mathcal{A} \subset \mathcal{X} \otimes \mathcal{Y}$ can be perfectly discriminated by separable measurements in the case $\mathcal{X} = \mathcal{Y} = \mathbb{C}^3$ [DMS+03].

In this work, we provide an easily checkable characterization of when an unextendable product set is perfectly discriminated by separable measurements, and we use this characterization to present the first example of an unextendable product set in $\mathcal{X} \otimes \mathcal{Y}$, for $\mathcal{X} = \mathcal{Y} = \mathbb{C}^4$, that is not perfectly discriminated by separable measurements. This resolves an open question raised in [DFXY09].

We also show that every unextendable product set together with one extra pure state orthogonal to every member of the unextendable product set is not perfectly discriminated by separable measurements. In order to prove this, we observe a connection with a family of positive maps previously studied in the literature [Ter01, BGR05].

An optimal bound on discriminating the Yu–Duan–Ying states

A set of four locally indistinguishable maximally entangled states in $\mathbb{C}^4 \otimes \mathbb{C}^4$ was exhibited by Yu, Duan and Ying in [YDY12]. Before this example, it was not known whether there existed such sets of $N \leq n$ maximally entangled states in $\mathbb{C}^n \otimes \mathbb{C}^n$. It was shown in [Cos13] that the maximum probability of distinguishing the Yu–Duan–Ying set, by means of PPT measurements, is at most 7/8. In this work, we prove that the maximum success probability for any separable measurement to distinguish the same set is 3/4, assuming a uniform selection of states. This bound is easily seen to be achievable by an LOCC measurement, implying that it is the optimal success probability of an LOCC measurement for this problem. The upper bound is closely connected to the positive maps of Breuer and Hall [Bre06, Hal06].

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