Tensor-stable positive maps for quantum information theory

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We consider here positive maps $\mathcal{P} : \mathcal{M}_d \to \mathcal{M}_{d'}$, i.e. mapping complex $d \times d$-matrices linearly to complex $d' \times d'$-matrices in such a way that positive matrices are mapped to positive matrices. For any given $n \in \mathbb{N}$, a positive map $\mathcal{P}$ is called n-**tensor-stable positive** if the n-fold parallel application $\mathcal{P}^{\otimes n}$ is a positive map. A map $\mathcal{P}$ is called **tensor-stable positive** if it is n-tensor-stable positive for all $n \in \mathbb{N}$.

There are two classes of tensor-stable positive maps occuring naturally in quantum information theory: **completely positive maps** and **completely co-positive maps** (the latter are maps of the form $\mathcal{T} \circ \vartheta$ with a completely positive map $\mathcal{T} : \mathcal{M}_d \to \mathcal{M}_{d'}$ and where $\vartheta : \mathcal{M}_d \to \mathcal{M}_d$ is the usual matrix transposition). In the following, we will denote these two classes by **trivial** tensor-stable positive maps.

We study the question of whether there exist non-trivial tensor-stable positive maps. Such maps would have profound impact on quantum information theory: *(i)* they would prove the existence of NPT-bound entanglement [7, 3, 2]; *(ii)* they would provide new families of quantum channels with vanishing quantum capacity (cf. [9]).

In this paper, we first show the existence of non-trivial n-tensor-stable positive maps for any given $n \in \mathbb{N}$. Our further main results follow from connecting, in a quantitative way, the property of n-tensor-stable positivity with the property of entanglement distillability. This relies on a generalization of distilliation and coding procedures to block-positive matrices (i.e. to entanglement witnesses). Applying this method in the limit $n \to \infty$, we prove the connection to the NPT-bound entanglement problem mentioned above, and the statement that any tensor-stable positive map acting on a qubit domain is necessarily trivial.

Results and used methods

Using unextendible product bases [1], we prove the following:

Theorem 1. *For any $n \in \mathbb{N}$ and any $d, d' \geq 2$, there exists a non-trivial n-tensor-stable positive map $\mathcal{P} : \mathcal{M}_d \to \mathcal{M}_{d'}$.*

Unfortunately the techniques used to prove this lemma do not provide a non-trivial tensor-stable positive map, i.e. a map which remains n-tensor-stable positive for all $n \in \mathbb{N}$.

Our main method for studying the existence problem of tensor-stable positive maps is a generalization of distillation techniques. Therefore consider the Choi-matrix $C_{\mathcal{P}} := (\mathrm{id}_d \otimes \mathcal{P})(\omega_d)$ of a positive map $\mathcal{P} : \mathcal{M}_d \to \mathcal{M}_{d'}$. It is well-known that such a matrix is block-positive, i.e.

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$\langle\psi\phi|C_{\mathcal{P}}|\psi\phi\rangle \geq 0$ for all product vectors $|\psi\phi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^{d'}$. We will use the ω-**negativity** defined as

$$\nu_\omega(\mathcal{P}) := \|C_{\mathcal{P}}\|_1 - \operatorname{tr}(C_{\mathcal{P}}) \tag{1}$$

to quantify the distance of $\mathcal{P}$ to the cone of completely positive maps.

We have the following lemma:

Lemma 1. *Let $\mathcal{P} : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}$ be a n-tensor-stable positive map. Then:*

$$\frac{\nu_\omega(\mathcal{P})}{\|\mathcal{P}\|_\diamond} \leq \inf_{\mathcal{S}} \|\omega_d - \mathcal{S}\left(C_{\mathcal{P}}^{\otimes(n-1)}\right)\|_1, \tag{2}$$

where the infimum is taken over all separable completely positive maps $\mathcal{S} : \mathcal{M}_{d_1} \otimes \mathcal{M}_{d_2} \to \mathcal{M}_d \otimes \mathcal{M}_d$, i.e. completely positive maps with product Kraus operators.

Lemma 1 connects the distance of an n-tensor-stable positive map from the completely positive maps to a generalized $(n-1)$-distillability of its Choi-matrix. Together with the following Lemma we can use results from the theory of entanglement distillation to obtain results for tensor-stable positive maps:

Lemma 2. *For a positive map $\mathcal{P} : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}$ we have:*

(i) If $\mathcal{P}$ is not completely co-positive, then there exists a separable completely positive map $\mathcal{S} : \mathcal{M}_{d_1 d_2} \to \mathcal{M}_{d^2}$ with

$$\mathcal{S}(C_{\mathcal{P}}) = \frac{1}{d^2-1}\left[\left(1-\frac{p}{d}\right)\mathbb{1}\otimes\mathbb{1} - \left(\frac{1}{d}-p\right)\mathbb{F}\right]. \tag{3}$$

for $p < 0$ and $d \in \{d_1, d_2\}$, i.e. we obtain an entangled Werner state [10].

(ii) If $\Gamma \circ \mathcal{P}$ or $\mathcal{P} \circ \Gamma$ is not completely positive for the reduction map [6] defined by $\Gamma(X) := \operatorname{tr}(X)\mathbb{1} - X$, then there exists a separable completely positive map $\mathcal{S} : \mathcal{M}_{d_1 d_2} \to \mathcal{M}_{d^2}$ with

$$\mathcal{S}(C_{\mathcal{P}}) = \frac{1}{d^2-1}\left[(1-p)\,\mathbb{1}\otimes\mathbb{1} - (1-dp)\,\omega_d\right] \tag{4}$$

for $p > \frac{1}{d}$ and $d \in \{d_1, d_2\}$, i.e. we obtain an entangled isotropic state.

Lemma 2 is proved via a standard twirling argument [6] generalized to block-positive matrices. Together with Lemma 1 we obtain some important results. As all entangled qubit Werner states are distillable [2] and therefore lead to a separable completely positive map $\mathcal{S}$ needed for the infimum in Lemma 1, we obtain:

Theorem 2 (No tensor-stable positive qubit maps)**.** *There are* no *non-trivial tensor-stable positive maps $\mathcal{T} : \mathcal{M}_2 \to \mathcal{M}_d$ or $\mathcal{T} : \mathcal{M}_d \to \mathcal{M}_2$ for any $d \in \mathbb{N}$.*

Furthermore, if *all* entangled Werner states were distillable we would obtain a separable completely positive map needed for Lemma 1 for any tensor-stable positive and not completely co-positive map. Thus we get:

Theorem 3 (Tensor-stable positivity implies NPT-bound entanglement)**.** *If there exists a non-trivial tensor-stable positive map $\mathcal{T} : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}$, then there exist NPT bound-entangled states in dimensions $\mathcal{M}_{d_1} \otimes \mathcal{M}_{d_1}$ and $\mathcal{M}_{d_2} \otimes \mathcal{M}_{d_2}$.*

Finally we provide a one-parameter family of positive maps containing a tensor-stable positive map, iff such a map exists. Similarly to the NPT-bound entanglement question, the existence of a nontrivial tensor-stable positive map can thus be decided by a one-parameter family of maps:

Theorem 4 (One parameter family of candidates for tensor-stable positivity)**.** *There exists a non-trivial tensor-stable positive map $\mathcal{P} : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}$ iff there is $p < 0$ and $d \in \{d_1, d_2\}$ such that*

$$\mathcal{P}_\beta := \mathcal{P}_W^{(p)} \otimes \vartheta \circ \mathcal{P}_W^{(p)} \tag{5}$$

is tensor-stable positive, where $\mathcal{P}_W^{(p)} : \mathcal{M}_d \to \mathcal{M}_d$ denotes the map corresponding to the Werner state with parameter $p \in [-1, 1]$ via the Choi-Jamiolkowski isomorphism.

Applications of tensor-stable positive maps

Tensor-stable positive maps are connected to other interesting questions in quantum information theory.

The existence of non-trivial tensor-stable positive maps would have implications on capacity bounds for quantum channels. For a quantum channel $\mathcal{T} : \mathcal{M}_d \to \mathcal{M}_d$, let $\mathcal{Q}_2(\mathcal{T})$ denote the quantum capacity assisted by 2-way classical communication and let $\mathcal{Q}(\mathcal{T})$ denote the quantum capacity of $\mathcal{T}$.

Theorem 5. *Let $\mathcal{P} : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}$ be a non-trivial tensor-stable positive map and $\mathcal{T} : \mathcal{M}_{d_1} \to \mathcal{M}_{d_1}$ be a quantum channel.*

(i) If $\mathcal{P} \circ \mathcal{T}$ is a quantum channel, then $\mathcal{Q}_2(\mathcal{P} \circ \mathcal{T}) = 0$.

(ii) If $d_1 = d_2$ and $\mathcal{P}$ is invertible and unital, then $\mathcal{Q}(\mathcal{T}) \leq \frac{log\left(\|\mathcal{P}^{-1} \circ \mathcal{T}\|_\diamond \|\mathcal{P}^\star(\mathbb{1})\|_\infty\right) log(d_1)}{log(\|\vartheta \circ \mathcal{P}^\star \circ \vartheta\|_\diamond)}$.

Thus, the class of quantum channels $\mathcal{T}$ for which $\mathcal{P} \circ \mathcal{T}$ is a quantum channel would provide a new family of channels with vanishing capacity. This is analogous to the class of completely co-positive channels $\mathcal{T}$, which have vanishing quantum capacity.

Secondly, in [8, 4, 5] the authors ask, whether there exist quantum channels $\mathcal{T} : \mathcal{M}_d \to \mathcal{M}_d$ which are not entanglement breaking, but which is still such that for all $n \in \mathbb{N}$ the channel $\mathcal{T}^{\otimes n}$ maps any input state to a separable state. They call such a channel ∞-locally entanglement annihilating. We show the following:

Theorem 6. *If there exists a ∞-locally entanglement annihilating channel which is not entanglement breaking, then there exist a non-trivial tensor-stable positive map.*

Therefore the existence of such ∞-locally entanglement annihilating channels would again imply the existence of NPT-bound entanglement.

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