

An optimal adiabatic quantum query algorithm

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The quantum adversary method was originally introduced by Ambainis [Amb02] for lower-bounding the *quantum query complexity* $Q(f)$ of a function f . It is based on optimizing a matrix Γ assigning weights to pairs of inputs. It was later shown by H oyter et al. [HLS07] that using negative weights also provides a lower bound, which is stronger for some functions. A series of works [RS12, Rei09, Rei11] then led to the breakthrough result that this *generalized* adversary bound, which we will simply call adversary bound from now on, actually characterizes the quantum query complexity of any function f with boolean output and binary input alphabet. This is shown by constructing a tight algorithm based on the dual of the semidefinite program corresponding to the adversary bound. Finally, Lee et al. [LMR⁺11] have closed the question for all functions, in generalizing this result to the quantum query complexity of state conversion, where instead of computing a function $f(x)$, one needs to convert a quantum state $|\rho_x\rangle$ to another quantum state $|\sigma_x\rangle$.

All these results were obtained in the usual discrete-time query model, where each query corresponds to applying a unitary oracle O_x . In this model, an algorithm then consists in a series of input-independent unitaries U_1, U_2, \dots, U_T , interleaved with oracle calls O_x . Another natural model is the continuous-time model, or Hamiltonian-based model, where the oracle corresponds to a Hamiltonian H_x , and the algorithm consists in applying a possibly time-dependent, but input-independent, *driver* Hamiltonian $H_D(t)$, together with the oracle Hamiltonian. The two models are related by the fact that the unitary oracle O_x can be simulated by applying the Hamiltonian oracle H_x for some constant amount of time. This implies that the continuous-time model is at least as powerful as the discrete-time model. In the other direction, Cleve et al. [CGM⁺09] have shown that the discrete-time model can simulate the continuous-time model up to at most a sublogarithmic overhead, which implies that the continuous- and discrete-time models are equivalent up to a sublogarithmic factor. Lee et al. [LMR⁺11] later improved this result to a full equivalence of both models, by showing that the fractional query model, an intermediate model proved in [CGM⁺09] to be equivalent to the continuous-time model, is also lower bounded by the adversary bound, so that all these models are characterized by this same bound (in the case of functions, a similar result can be obtained by extending an earlier proof of Yonge-Mallo, originally considering the adversary bound with positive weights, to the case of negative weights [YM11]).

Even though these results imply that the continuous-time quantum query complexity is characterized by the adversary bound, they do not provide an explicit Hamiltonian-based query algorithm, except the one obtained from the discrete-time algorithm by replacing each unitary oracle call by the application of the Hamiltonian oracle for a constant amount of time. The resulting Hamiltonian of this algorithm then involves many discontinuities (at all times in between unitary gates), which is not very satisfying from the point of view of physics, where *reasonable* Hamiltonians are smooth. However, such discontinuities are not unavoidable, as for some problems, continuous-time

query algorithms based on smooth Hamiltonians are known. The first example is unstructured search, for which Farhi and Gutmann [FG96] proposed a continuous-time analogue of Grover’s algorithm based on a simple time-independent Hamiltonian (later, Roland and Cerf [RC02] also proposed an adiabatic version of this algorithm, based on a slowly varying Hamiltonian). Later, Farhi et al. [FGG08] proposed a quantum algorithm for the NAND-tree based on scattering a wave incoming on the tree, also using a time-independent Hamiltonian. It is precisely this algorithm that, through successive extensions, led to the tight algorithm based on the adversary bound for any function in [Rei11], but these extensions were using the discrete-time model.

Our main contributions are to give a new continuous-time quantum query algorithm for any state conversion algorithm based on a slowly varying Hamiltonian, and also provide a direct proof of its optimality based on Ehrenfest’s theorem. The correctness of this algorithm relies on a spectra lemma leading to a rigorous proof of the quantum adiabatic theorem for this Hamiltonian. This implies that the continuous-time quantum query complexity of any state conversion problem is characterized by the adversary bound.

All technical details can be found in the full version of this article [BR14].

1 Adversary lower bound in the continuous-time model

We consider quantum state conversion problems, where the goal is to convert a state $|\rho_x\rangle$ taken from a set $\{|\rho_y\rangle : y \in A\}$ into the corresponding state $|\sigma_x\rangle$ in the set $\{|\sigma_y\rangle : y \in A\}$, the input $x \in X \subset \Sigma^n$ being accessible via a black box (Σ is a finite set). Since unitaries independent of the input x are free in the quantum query complexity model, such problems are completely defined by the Gram matrices $\rho_{xy} = \langle \rho_x | \rho_y \rangle$ and $\sigma_{xy} = \langle \sigma_x | \sigma_y \rangle$. The adversary bound for the quantum state generation problem (ρ, σ) is defined as follows

Definition 1. [LMR⁺11, LR13](Adversary bound)

$$\begin{aligned} \text{Adv}^*(\rho, \sigma) &= \max_{\Gamma} \|\Gamma \circ (\rho - \sigma)\| && \text{subject to} && \forall j \in [n], \quad \|\Gamma \circ \Delta_j\| \leq 1, \\ &= \gamma_2(\rho - \sigma | \Delta), \end{aligned}$$

$$\text{where } \Delta = \{\Delta_1, \dots, \Delta_n\}.$$

We first give a direct proof that this bound, originally considered for discrete-time quantum query complexity, is also a lower-bound in the continuous-time model.

Theorem 1.1. *For any $|A| \times |A|$ Gram matrices ρ, σ , we have*

$$\begin{aligned} Q_0^{\text{ct}}(\rho, \sigma) &\geq \frac{1}{2} \text{Adv}^*(\rho, \sigma) \\ Q_\epsilon^{\text{ct}}(\rho, \sigma) &\geq \frac{1}{2} \min_{\sigma' : \mathcal{F}_H(\sigma, \sigma') \geq \sqrt{1-\epsilon}} \text{Adv}^*(\rho, \sigma'). \end{aligned}$$

While our proof is unsurprisingly similar to the proof in the discrete-time case, one originality is that it uses Ehrenfest’s theorem [Ehr27], which expresses the evolution of the expectation value of an observable $\langle \Gamma \rangle_t$ in terms of its commutator with the Hamiltonian:

$$\frac{d\langle \Gamma \rangle_t}{dt} = \frac{1}{i} \langle [\Gamma, H(t)] \rangle_t + \left\langle \frac{\partial \Gamma}{\partial t} \right\rangle_t.$$

Indeed, we use the fact that the adversary matrix Γ is Hermitian and can therefore be considered as an observable measuring the progress of the algorithm toward the target state.

2 Adiabatic quantum query algorithm

We construct a new quantum query algorithm in the continuous-time model based on the adiabatic principle of Born and Fock [BF28].

A quantum system with a time-dependent Hamiltonian remains in its instantaneous eigenstate if the Hamiltonian variation is slow enough and there is a large gap between its eigenvalue and the rest of the spectrum of the Hamiltonian.

Our adiabatic quantum query algorithm, which we call **AdiaConvert** $(\rho, \sigma, \varepsilon)$, is extremely simple. Its slowly varying Hamiltonian consists in a difference of two projectors, one being the (time-independent) oracle Hamiltonian Π_x , and the other being a slowly varying driver Hamiltonian $\Lambda(s, \varepsilon)$ projecting on the vector space spanned by a set of states $\{|\Psi_x^-(s, \varepsilon)\rangle \mid x \in A\}$. These states, derived from the dual expression of the adversary bound, are such that for $s \in [0, 1]$, the 0-eigenstate of the total Hamiltonian $H_x(s)$ simply interpolates (up to a small error) between the initial state $|0, \rho_x\rangle$ and the target state $|1, \sigma_x\rangle$ (an additional qubit is introduced to make these states orthogonal). We therefore obtain the following:

Proposition 1. *For any state conversion problem (ρ, σ) and error ε such that $\text{Adv}^*(\rho, \sigma) \geq \varepsilon$, the algorithm **AdiaConvert** $(\rho, \sigma, \varepsilon)$ converts $|0, \rho_x\rangle$ (for any $x \in A$) into a state ε -distant to $|1, \sigma_x\rangle$ in time $T = O(\text{Adv}^*(\rho, \sigma)/\varepsilon^2)$.*

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