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Cellular-automaton decoders for topological quantum memories

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Decoders are one of the necessary building blocks for robust local quantum memories, basic primitives that must arguably be part of any functioning topological quantum computer. While surface codes have raised the prospects for scalability and performance of quantum storage devices, the task of decoding resorts to complicated classical algorithms. Such decoding algorithms are typically designed for traditional centralized computing architectures. This questions whether active decoding can be fast enough to outpace the decoherence time in systems with many qubits, giving rise to significant challenges. In this work, we address this question by recasting error recovery as a dynamical process on a field generating cellular automaton. We envisage quantum systems controlled by a classical hardware composed of small local memories, communicating with nearest neighbors only, and repeatedly performing identical simple update rules. This framework for constructing topological quantum memories does not require any global operations or complex decoding algorithms. Furthermore, the local updates rules do not have to be perfect or synchronized, relaxing many of the former requirements on decoding devices. Hence, our work raises prospects for the technical feasibility of scalable and ultra-fast decoding devices.

Through comprehensive numerical study (see Fig. 1c) we find that equipping error syndromes (anyons) with an attractive scalar field of the form $1/r^{\alpha}$, can provide a working decoder with an error correction threshold and exponential error suppression. Considering the anyons to move according to the gradient of this field, we give an argument why error correction is possible only if $\alpha \geq 1$, i.e. the field is not too long ranged. A key insight is that in a discretized approximation of such fields can be generated via simple and local cellular automaton update rules in an efficient and robust manner. The field takes real values on a periodic D-dimensional square lattice $\mathbb{Z}_L^D$, with anyons constrained to a two-dimensional plane. At time t and location $\boldsymbol{x} \in \mathbb{Z}_L^D$ the field has a value $\phi_t(\boldsymbol{x})$, and at later times updates according to

$$\phi_{t+1}(\boldsymbol{x}) = \frac{\phi_t(\boldsymbol{x})}{2} + \frac{1}{4D} \sum_{\langle \boldsymbol{y},\boldsymbol{x} \rangle} \phi_t(\boldsymbol{y}) + q_E(\boldsymbol{x}). \tag{1}$$

The second term is a sum of all neighboring cells, and the last term is charge-like with unit value when an anyon is present and zero otherwise. We prove that for a fixed anyon distribution $q_E(\boldsymbol{x})$ the field converges to a unique fixed point $\tilde{\phi}_\infty$, up to an unimportant gauge.

For $D = 3$ the field approximates the desired $1/r$ scaling. The rate of convergence is captured by two statements. First, we show $\|\tilde{\phi}_t - \tilde{\phi}_\infty\|_2 \leq \mathrm{e}^{-(\pi^2/2D)t/L^2} \|\tilde{\phi}_0 - \tilde{\phi}_\infty\|_2$, where $\|\cdot\|_2$ is the Euclidean norm and the tilde merely indicates a gauge fixing of the field. This bound leads to a convergence time which is proportional to L^2. This can be understood as the requirement for information propagate distances compared to the lattice size to ensure global convergence, and propagation is diffusive. However, percolation arguments show that the effective size of error strings will be at most of the order $\log(L)$ and we also know that local, pointwise, convergence is a more relevant quantity. Our second statement shows the point-wise convergence

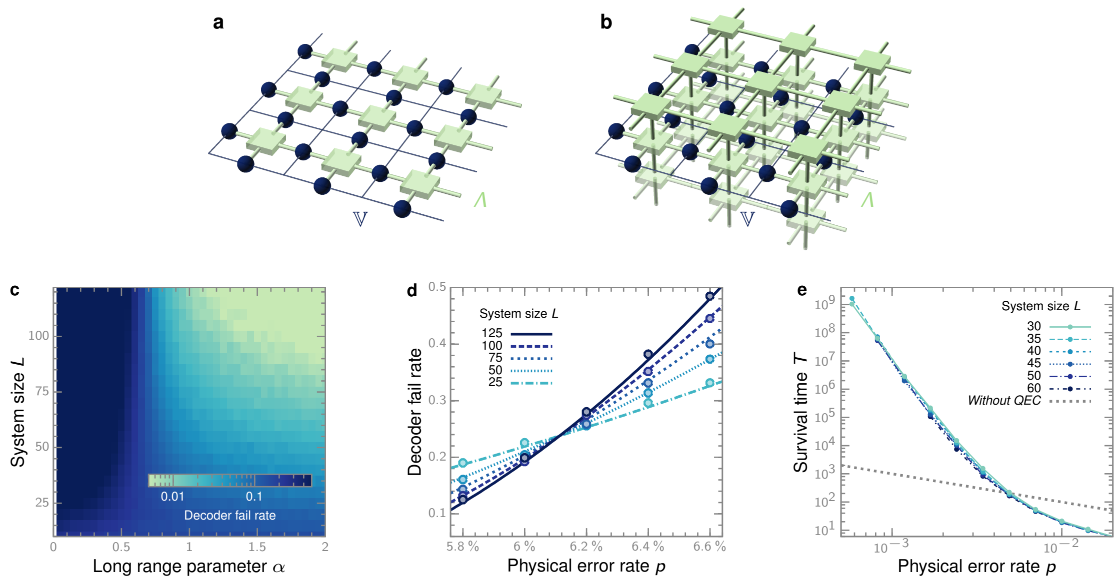

FIG. 1. (**a**) The toric code on a periodic lattice controlled by a 2D classical cellular automaton decoder. Blue spheres represent the physical qubits and the green boxes represent the elementary cells of the automaton. The communication between cells and with the syndrome measurements (anyons) is nearest neighbor, indicated via connecting tubes. (**b**) A 3D cellular automaton decoder with the central layer hosting the toric code. (**c**) Numerical evidence for a phase transition from the non-decoding to the decoding regime in the long range parameter α of the $1/r^{\alpha}$ field. (**d**) Threshold for the 3D cellular automaton decoder, assuming perfect measurements. (**e**) First numerical evidence of a high noise threshold in the dynamical setting, where we apply the update rules continuously, hence preventing the creation and diffusion of anyon pairs.

of the field over the distance corresponding to the maximal length of errors strings, following an anyon hopping by a unit vector $\boldsymbol{e}$. It can be stated formally as saying that for every $\epsilon > 0$, $|\phi_t(\boldsymbol{x}) - \phi_\infty(\boldsymbol{x})| < \epsilon$ holds whenever

$$t > t_{\text{min}} \leq \frac{4\pi^2(D-1)}{\mathrm{e}D^2}\,(D\|2\boldsymbol{x}+\boldsymbol{e}\|_1)^{\frac{2}{D-1}}\;\epsilon^{\frac{2}{1-D}}, \tag{2}$$

assuming that ϕ_0 corresponds to the converged field prior to an anyon hop from position $\boldsymbol{x}$ to $\boldsymbol{x}+\boldsymbol{e}$.

As an immediate consequence of the bound in Eq. (2) we obtain for $D = 3$ that whenever $t > t_{\text{min}} \leq \kappa \log^2(L)$ for some constant κ, sufficient equilibration of the field between anyon movements is guaranteed. This bound comes from identifying $\epsilon \sim 1/\log^2(L)$, so the error is smaller than the field gradient in clusters of errors that need to be corrected. This implies, that the anyons move as if the field were equilibrated with the desired power-law potential.

Our cellular automata decoders work by a repeated sequence of c iterations of the field update followed by a single anyon update. When performing an anyon update, anyons probabilistically hop to the adjacent cell with largest field value. In Ref. [1], we study the resulting 3D cellular automaton decoder when measurements are perfect and performed only after all anyons are removed. The deduced 3D decoder has an error threshold of $6.1\,\%$. A similar algorithm on a 2D lattice, but with time evolving c, takes advantage of non-equilibrium dynamics to give a threshold of $8.2\,\%$.

The working principle of our decoders closely resembles fundamental physical laws. In fact, it corresponds to an attractive interaction between anyonic excitations stabilizing a topological state, one reminding of gravity or electrostatics. Thus our work is a first step in connecting decoders with the design of naturally noise resistant quantum memories. In this mindset it is particularly interesting that our 3D-field decoder can also continuously counteract the creation of errors. This transfers the study of decoders from the pure algorithmic viewpoint to the realm of interacting particle systems. Only in this *dynamical setting*, it is possible to consider literal error rates and more distinct error models like thermal or one sided noise. Further unpublished numerical findings provide the first evidence for a threshold error rate of $\sim 5 \times 10^{-4}$ in the dynamical setting. This extends the single previous result given in Ref. [2], which proves a threshold error rate of 2.4×10^{-11}. The accompanying numerics in Ref. [2] indicate the possibility of an error threshold with the same order of magnitude as ours. Likewise we confirm that the consideration of measurement errors has only a slight impact on the error threshold, and does not require any adjustments on the actual algorithm.

Gács' robust classical memory in one dimension [3, 4] suggests that a dynamical decoder with constant local overhead for the toric code can exist in principle. So far every known proposal inherits some additional logarithmic overhead in the number of physical qubits. This overhead seems to be necessary if one wants to avoid implementing a *self-simulating* cellular automaton, which require update rules capable of performing universal computation [3]. The same applies for our decoder in the form of a $\log(L)$ dependence on the speed of the local classical processing, and another $\log(L)$ local memory size. Our scheme does not require any complicated message passing system, nor does it build upon a hierarchical structure, and many of the previous parallelized decoders. Rather, all of the decoding information is encoded in a single memoryless scalar field. An appealing aspect of our proposal is its conceptual simplicity which allows us to provide clear estimates on the required resources.

To conclude we discuss the possible further implications of our work. Clearly our proposal has immediate impact on the actual realization of decoders. The competitive error thresholds combined with the simplicity, robustness, and efficiency of our local rules are most suited for a direct implementation in specialized hardware. This indicates the feasibility of scalable and ultra-fast decoding devices. Our design directly contributes to the recent question whether decoding can be integrated into hardware at all [5], which is necessary in order to keep pace with further experimental developments [6, 7].

Perhaps even more important, our proposal provides a framework for analyzing topological code decoders in the light of cellular automata. Up to the knowledge of the authors, the work at hand constitutes the first decoder for a topological code which can operate in continuous time; which in the cellular automaton language corresponds to asynchronous updates of the rules. This enables an honest benchmark of our decoder, incorporating the real temporal evolution of errors and allowing for more distinct error models. Important classical results like Toom's stability theorem [8] or the disproof of the positive rates conjecture [3, 4] might be adapted to this setting. We expect that such analytical insights will have further fundamental implications on the design of topological quantum memories.

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