On a new Quantum Rényi Divergence

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[J. Math. Phys. 54, 122203 (2013)]

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[arXiv:1306.1586]

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[arXiv:1309.3228]

Outline

- Motivation and Contributions
- Definition
- Elementary Properties
- Operational Interpretation (hypothesis testing)
- Application to Strong Converse (classical data over entanglement-breaking channels)
- Conditional Quantum Rényi Entropy

Motivation: QIP 2014 Talks and Posters using this Quantum Rényi Divergence

12:20h **The second laws of quantum thermodynamics**
Fernando Brandao, Michal Horodecki, **Jonathan Oppenheim** , Nelly Ng, Stephanie Wehner

09:30h **Strong converses for quantum channel capacities**
Andreas Winter

11:50h **Entanglement sampling and applications**
Frédéric Dupuis, **Omar Fawzi** , Stephanie Wehner

11:20h **Robust protocols for securely expanding randomness and distributing keys using untrusted quantum devices**
Carl Miller, Yaoyun Shi

Salman Beigi. Sandwiched Renyi Divergence Satisfies Data Processing Inequality

Koenraad Audenaert, Nilanjana Datta and Felix Leditzky. A limit of the quantum Renyi divergence

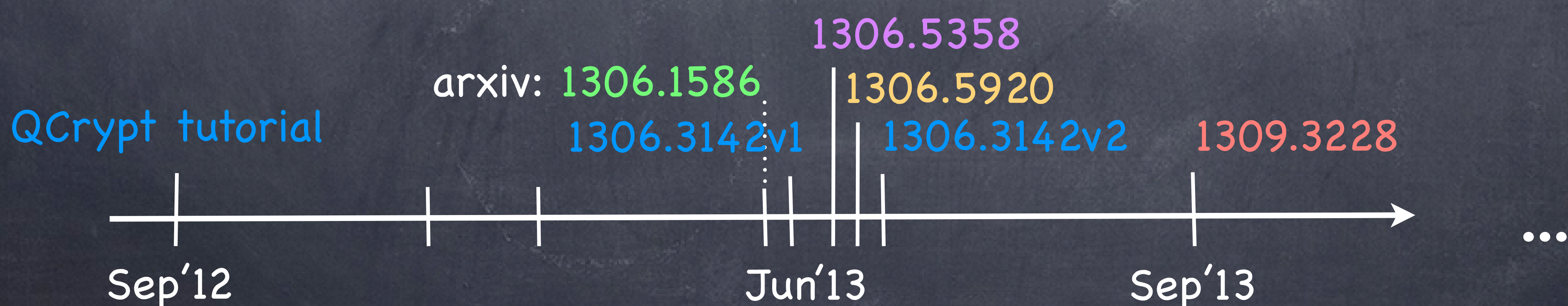
Milan Mosonyi. Renyi divergences and the classical capacity of finite compound channels

Manish Gupta and Mark Wilde. Multiplicativity of completely bounded p-norms implies a strong converse for entanglement-assisted capacity

Patrick Coles and Fabian Furrer. Entropic Error-Disturbance Relations

Contributions

- New Rényi conditional entropy proposed, specializes to known quantities (Team Müller-Lennert)
- Independent discovery of the divergence as a proof tool for strong converse (Team Wilde)
- Properties and conjectures released (Team Blue)
- All conjectures resolved (Frank&Lieb, Beigi, Team Blue)
- Operational interpretation in hypothesis testing discovered (Team Mosonyi)



Classical Rényi Divergence

- Given two probability distributions P and Q:

$$D_\alpha(P\|Q) := \frac{1}{\alpha - 1} \log\left(\sum_x P(x)^\alpha Q(x)^{1-\alpha}\right)$$

- Operational significance in information theory, for example in the study of error exponents and cutoff rates.

- Versatile tool in proofs, for example to derive one-shot bounds.

Quantum Rényi Relative Entropy

- Petz investigated quantum generalizations of Csiszár's f-divergences.
- For two quantum states ρ and σ :

$$D_\alpha(\rho\|\sigma) := \frac{1}{\alpha - 1} \log \operatorname{tr}\left(\rho^\alpha \sigma^{1-\alpha}\right)$$

- Desirable properties in the range $\alpha \in [0, 2]$ due to the operator concavity/convexity of $f : t \mapsto t^\alpha$
- Operational significance in the direct part of quantum hypothesis testing

Quantum Rényi Divergence / Sandwiched Rényi Divergence

- For $\alpha \in (0,1) \cup (1,\infty)$ and $\rho, \sigma \geq 0, \rho \neq 0$,

$$\boxed{\widetilde{D}_\alpha(\rho\|\sigma) := \frac{1}{\alpha - 1} \log \frac{\operatorname{tr}\left(\left(\sigma^{\frac{1-\alpha}{2\alpha}} \rho \, \sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha\right)}{\operatorname{tr}(\rho)}}$$

- For $\alpha > 1$ and if σ is not invertible, set $\sigma' = \sigma + \xi \mathbb{I}$ and take the limit $\xi \to 0$.
  - generalized inverse if $\sigma \gg \rho$
  - $+\infty$ otherwise

Quantum Rényi Divergence / Sandwiched Rényi Divergence

- For the following: $\mathrm{tr}(\rho) = 1$
- Divergence is related to the Schatten norm

$$\widetilde{D}_\alpha(\rho\|\sigma) = \frac{\alpha}{\alpha - 1} \log \left\| \sigma^{\frac{1-\alpha}{2\alpha}} \rho \, \sigma^{\frac{1-\alpha}{2\alpha}} \right\|_\alpha$$

$$\frac{\alpha}{1-\alpha} \log \Big\|$$

$$\Big\|_\alpha$$

Two-Parameter Family

- Both definitions can be seen as special cases of a two-parameter family of divergences:
  (Jaksic et al. / Audenaert&Datta)

$$\bar{D}_{\alpha,z}(\rho\|\sigma) := \frac{1}{\alpha-1}\log\operatorname{tr}\left(\left(\sigma^{\frac{1-\alpha}{2z}}\,\rho^{\frac{\alpha}{z}}\,\sigma^{\frac{1-\alpha}{2z}}\right)^{z}\right)$$

- Clearly, $\bar{D}_{\alpha,1} \equiv D_\alpha$ and $\bar{D}_{\alpha,\alpha} \equiv \widetilde{D}_\alpha$.

Limits and Special Cases of

$$\widetilde{D}_\alpha(\rho\|\sigma) := \frac{1}{\alpha-1}\log\operatorname{tr}\left(\left(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\,\sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha\right)$$

- Quantum Relative Entropy, $\alpha \to 1$:

$$D(\rho\|\sigma) = \widetilde{D}_1(\rho\|\sigma) = \operatorname{tr}\left(\rho(\log\rho - \log\sigma)\right)$$

- Datta's Max Relative Entropy, $\alpha \to \infty$:

$$D_{\max}(\rho\|\sigma) = \widetilde{D}_\infty(\rho\|\sigma) = \inf\left\{\lambda \in \mathbb{R} \,\middle|\, \rho \leq \exp(\lambda)\sigma\right\}$$

- Fidelity: $\widetilde{D}_{\frac{1}{2}}(\rho\|\sigma) = 2\log F(\rho,\sigma)$

- Collision: $\widetilde{D}_2(\rho\|\sigma) = \log\operatorname{tr}\left(\rho\sigma^{-\frac{1}{2}}\rho\sigma^{-\frac{1}{2}}\right)$

Properties of

$$\widetilde{D}_\alpha(\rho\|\sigma) := \frac{1}{\alpha-1}\log\operatorname{tr}\left(\left(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\,\sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha\right)$$

- Continuous in ρ and σ
- Additivity: $\widetilde{D}_\alpha(\rho\otimes\tau\|\sigma\otimes\omega) = \widetilde{D}_\alpha(\rho\|\sigma) + \widetilde{D}_\alpha(\tau\|\omega)$
- Positive Definite: For two quantum states, $\widetilde{D}_\alpha(\rho\|\sigma) \geq 0$ with equality iff* $\rho = \sigma$.
- Monotonically non-decreasing in α (also: Beigi)
- Scaling: $\widetilde{D}_\alpha(a\rho\|b\sigma) = \widetilde{D}_\alpha(\rho\|\sigma) + \log\frac{a}{b}$
- *Domination: $\sigma' \geq \sigma$ implies $\widetilde{D}_\alpha(\rho\|\sigma') \leq \widetilde{D}_\alpha(\rho\|\sigma)$

Data-Processing Inequality

- For any completely positive trace-preserving map $\mathcal{E}$ and $\alpha \geq 1/2$, we have

$$\widetilde{D}_\alpha(\rho\|\sigma) \geq \widetilde{D}_\alpha\big(\mathcal{E}(\rho)\big\|\mathcal{E}(\sigma)\big)$$

- Müller-Lennert et al. / Wilde et al. : $\alpha \in [1, 2]$
- Frank+Lieb: $\alpha \geq 1/2$
- Beigi: $\alpha \geq 1$
- (Mosonyi+Ogawa: $\alpha \geq 1$)

Asymptotic Achievability ($\alpha > 1$)

- For any sequence of measurement maps $\mathcal{M}_n$ (quantum-to-classical channels), we have

$$\widetilde{D}_\alpha(\rho\|\sigma) \geq \frac{1}{n}\widetilde{D}_\alpha\big(\mathcal{M}_n(\rho^{\otimes n})\big\|\mathcal{M}_n(\sigma^{\otimes n})\big)$$

- There exists a sequence $\mathcal{M}_n^*$ such that

$$\widetilde{D}_\alpha(\rho\|\sigma) = \lim_{n\to\infty} \frac{1}{n}\widetilde{D}_\alpha\big(\mathcal{M}_n^*(\rho^{\otimes n})\big\|\mathcal{M}_n^*(\sigma^{\otimes n})\big)$$

- This allows to lift many properties from the classical domain.

Hypothesis Testing

- State discrimination using POVM $\{T, \mathbb{I} - T\}$

$$\alpha_n(T) := \mathrm{tr}\left(\rho^{\otimes n}(\mathbb{I} - T)\right)$$
$$\beta_n(T) := \mathrm{tr}\left(\sigma^{\otimes n} T\right)$$

- Critical rate (quantum Stein's Lemma):

$$\lim_{n\to\infty} \max\left\{ -\frac{1}{n}\log\beta_n(T) \,\middle|\, T : \alpha_n(T) \le \varepsilon \right\} = D(\rho\|\sigma)$$

- Error exponents: what happens to $\alpha_n(T)$ if

$-\dfrac{1}{n}\log\beta_n(T) \neq D(\rho\|\sigma)$?

Quantum Hoeffding Bound

- We are interested in the quantity

$$\alpha_{n,r} := \min\left\{\alpha_n(T) \,\middle|\, T : \beta_n(T) \leq \exp(-nr)\right\}$$

- Rate below critical rate (Hayashi'07/Nagaoka'06):

$$\lim_{n\to\infty} -\frac{1}{n}\log\alpha_{n,r} = \sup_{0<\alpha<1} \frac{\alpha-1}{\alpha}\Big(r - D_\alpha(\rho\|\sigma)\Big)$$

if $r < D(\rho\|\sigma)$

- Yields operational interpretation of "old" Rényi relative entropy

Strong Converse Regime

- If $r > D(\rho\|\sigma)$, we expect $1 - \alpha_{n,r}$ to drop exponentially in n.

- We show that

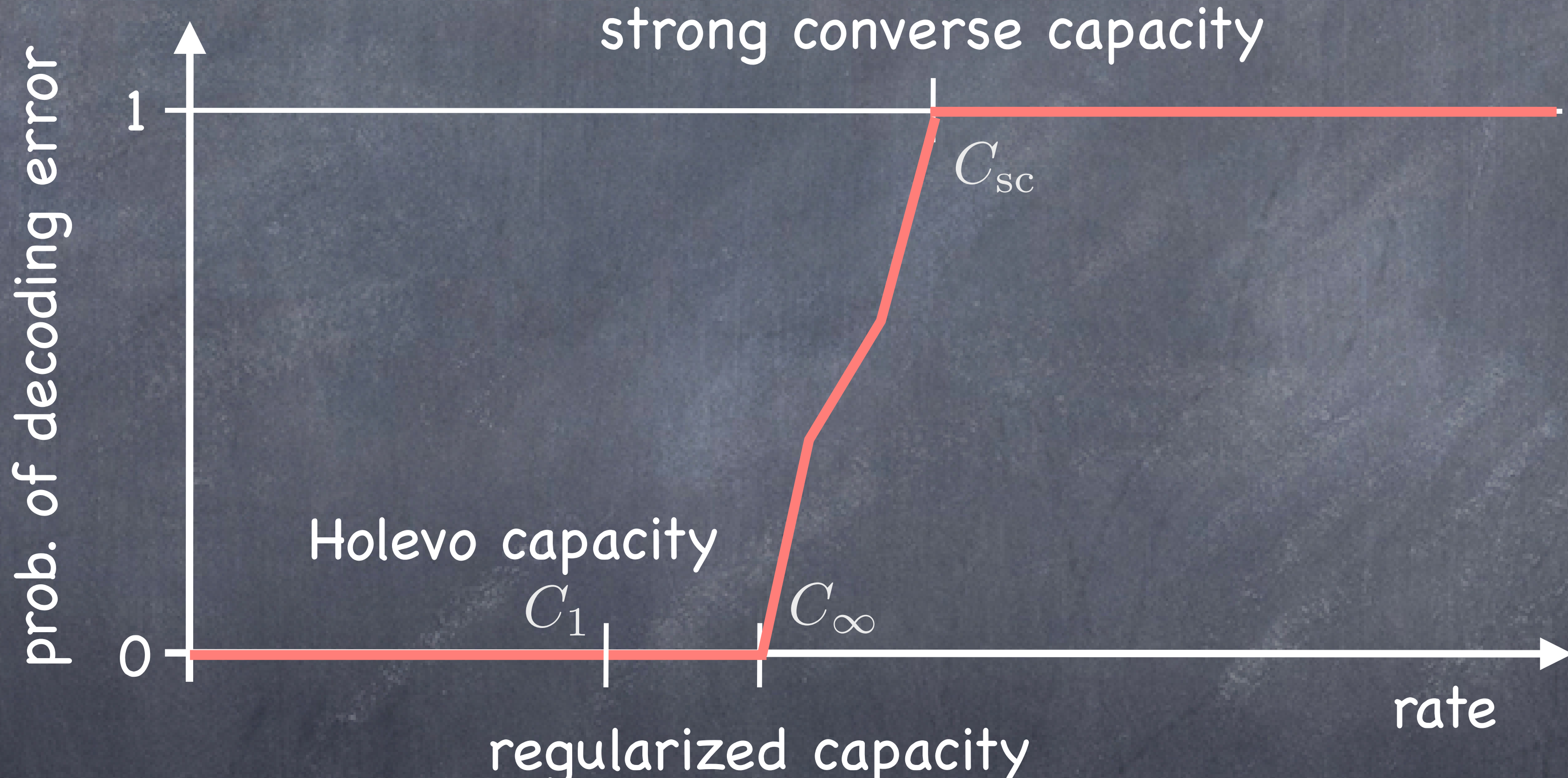
$$\lim_{n\to\infty} -\frac{1}{n}\log(1-\alpha_{n,r}) = \sup_{\alpha>1} \frac{\alpha-1}{\alpha}\Big(r - \widetilde{D}_\alpha(\rho\|\sigma)\Big)$$

- Yields an operational interpretation of the "new" Rényi divergence

- More direct interpretation (without optimization) using cut-off rates

Strong Converse Capacity

- Minimum of all R, such that every code with rate exceeding R leads to an asymptotically vanishing probability of successful decoding.



Entanglement-breaking (EB) and Hadamard channels: Holevo and regularized capacity agree

α-Information Radius

- For a single use of the channel, the success probability is bounded by, for any $\alpha > 1$,
(proof due to Polyanskiy&Verdú'10 / Sharma&Warsi'13)

$$p_{\mathrm{succ}} \leq M^{\frac{1-\alpha}{\alpha}} \exp\left(\frac{\alpha - 1}{\alpha} \widetilde{R}_\alpha(\mathcal{W}_{A\to B}) \right)$$

where M is the number of different messages, and the α-information radius is

$$\widetilde{\chi}_\alpha(\mathcal{W}_{A\to B}) := \inf_{\sigma_B} \sup_{\rho_A} \widetilde{D}_\alpha\big(\mathcal{W}_{A\to B}(\rho_A) \big\| \sigma_B\big) \,.$$

- Holevo capacity: $C_1 = \widetilde{\chi}_1(\mathcal{W}_{A\to B})$

Sub-Additvity ⇒ Strong Converse

- For n uses of the channel and $M = \exp(nR)$:

$$p_{\text{succ}} \leq \exp\left(\frac{\alpha - 1}{\alpha}\Big(\widetilde{\chi}_\alpha(\mathcal{W}_{A\to B}^{\otimes n}) - nR\Big)\right)$$

- Assume sub-additivity of information radius:

$$\widetilde{\chi}_\alpha(\mathcal{W}_{A\to B}^{\otimes n}) \leq n\widetilde{\chi}_\alpha(\mathcal{W}_{A\to B})$$

- For any $R > \widetilde{\chi}_1(\mathcal{W}_{A\to B})$, we find $\alpha > 1$ such that $R > \widetilde{\chi}_\alpha(\mathcal{W}_{A\to B})$ (by continuity)

- Thus, $p_{\text{succ}} \leq \exp\left(n\frac{\alpha - 1}{\alpha}(\widetilde{\chi}_\alpha(\mathcal{W}_{A\to B}) - R)\right)$

Sub-additivity ⇒ all three capacities agree

α-Information Radius for EB Channels is Sub-Additive

- Using the EB CPM $\widehat{\mathcal{W}}(\cdot) = \sigma^{\frac{1-\alpha}{2\alpha}} \mathcal{W}(\cdot) \sigma^{\frac{1-\alpha}{2\alpha}}$,

$$\widetilde{D}_\alpha(\mathcal{W}(\rho) \| \sigma) = \frac{\alpha}{\alpha - 1} \log \left\| \widehat{\mathcal{W}}(\rho) \right\|_\alpha$$

- Maximal output norm of EB CPM (with any CPM $\widehat{\mathcal{E}}$) is multiplicative (King'03):

$$\max_{\rho_{AA'}} \left\| (\widehat{\mathcal{E}} \otimes \widehat{\mathcal{W}})(\rho_{AA'}) \right\|_\alpha = \max_{\rho_A, \rho_{A'}} \left\| \widehat{\mathcal{E}}(\rho_A) \right\|_\alpha \cdot \left\| \widehat{\mathcal{W}}(\rho_{A'}) \right\|_\alpha$$

α-Information Radius for EB Channels is Sub-Additive (II)

$$
\begin{aligned}
\widetilde{\chi}_\alpha(\mathcal{E}\otimes\mathcal{W}) &= \min_{\sigma_{BB'}}\max_{\rho_{AA'}} \widetilde{D}_\alpha\big((\mathcal{E}\otimes\mathcal{W})(\rho_{AB})\big\|\sigma_{BB'}\big) \\
&\le \min_{\sigma_B,\sigma_{B'}}\max_{\rho_{AA'}} \widetilde{D}_\alpha\big((\mathcal{E}\otimes\mathcal{W})(\rho_{AB})\big\|\sigma_B\otimes\sigma_{B'}\big) \\
&= \min_{\sigma_B,\sigma_{B'}} \frac{\alpha}{\alpha-1}\log\max_{\rho_{AA'}}\big\|(\widehat{\mathcal{E}}\otimes\widehat{\mathcal{W}})(\rho_{AA'})\big\|_\alpha \\
&= \min_{\sigma_B,\sigma_{B'}} \frac{\alpha}{\alpha-1}\log\max_{\rho_A,\rho_{A'}}\big\|\widehat{\mathcal{E}}(\rho_A)\big\|_\alpha\big\|\widehat{\mathcal{W}}(\rho_{A'})\big\|_\alpha \\
&= \widetilde{\chi}_\alpha(\mathcal{E})+\widetilde{\chi}_\alpha(\mathcal{W})
\end{aligned}
$$

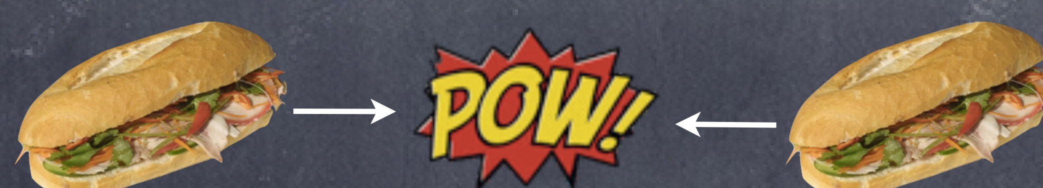
Conditional Rényi Entropy

- For a bipartite state ρ_{AB}, we define

$$\widetilde{H}_\alpha(A|B)_\rho := \sup_{\sigma_B} -\widetilde{D}_\alpha(\rho_{AB}\|\mathbb{I}_A \otimes \sigma_B).$$

- von Neumann Entropy: $H \equiv \widetilde{H}_1$
- Min-Entropy (Renner): $H_{\min} \equiv \widetilde{H}_\infty$ 

- Max-Entropy (König et al.): $H_{\max} \equiv \widetilde{H}_{\frac{1}{2}}$
- Collision Entropy (Renner): $\widetilde{H}_2$



- Classical case corresponds to Arimoto's conditional Rényi entropy

Duality and Uncertainty

- For a tripartite pure state ρ_{ABC}, we find

$$\widetilde{H}_\alpha(A|B) + \widetilde{H}_\beta(A|C) = 0, \quad \frac{1}{\alpha} + \frac{1}{\beta} = 2.$$

- Includes known relations for Min-/Max-Entropy and von Neumann entropy.

- Implies full side information generalization of Massen-Uffink uncertainty relations:

$$\widetilde{H}_\alpha(X|B) + \widetilde{H}_\beta(Y|C) \geq \log \frac{1}{c}$$

The proof is implied by properties of the conditional entropy together with a proof framework due to Coles et al.'12 (also Berta++'10, Tomamichel&Renner'11)