Strong, weak and pretty strong: Converses for quantum channel capacities

Andreas Winter (ICREA & UAB Barcelona)

1st (A)QIP: Aarhus 1998

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64 participants: whom do you know?

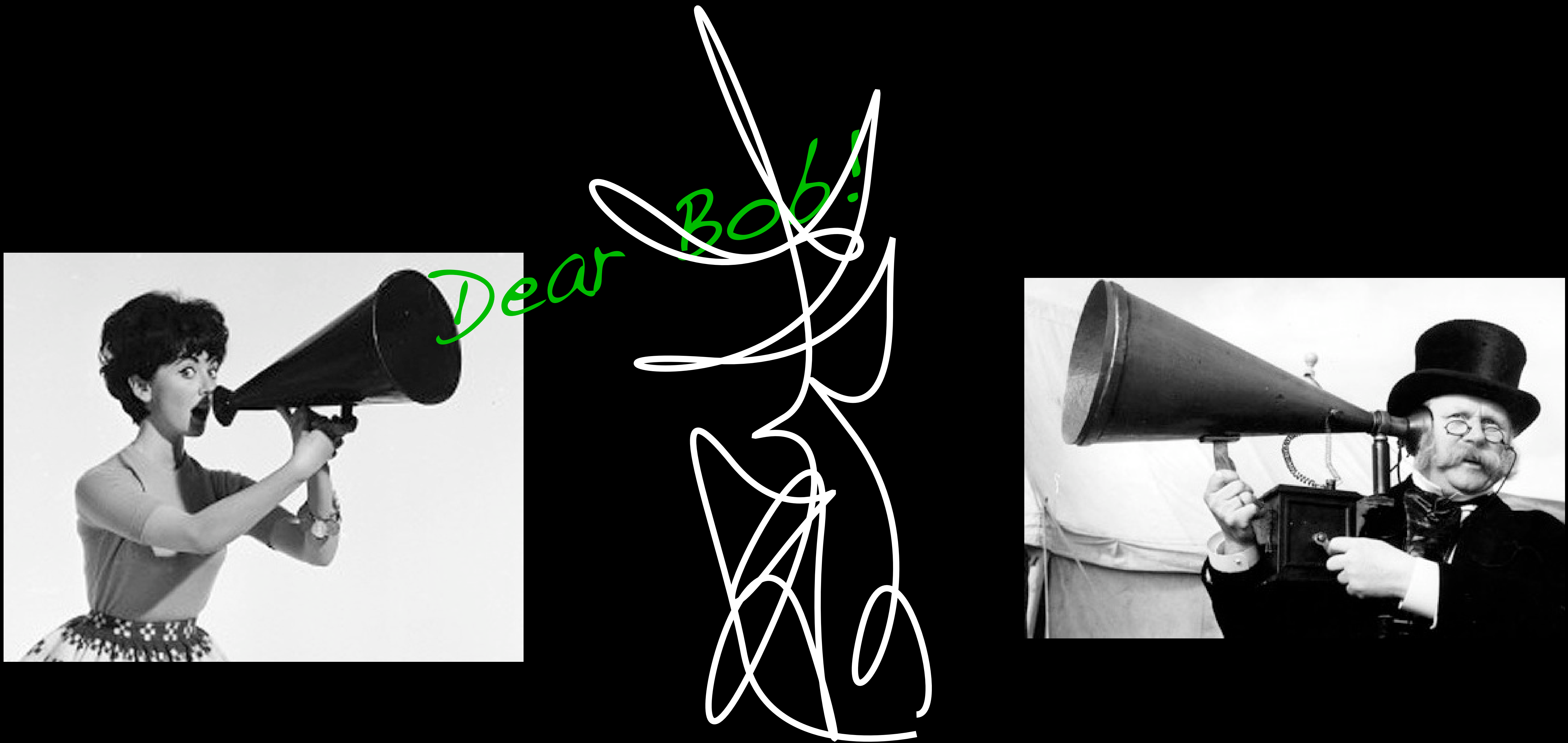
(circled: present @ QIP2014)

1. Communication

Dear Bob!

Shannon (1948): Fundamental problem is that of reproducing at one point a message selected at another point.

1. Communication

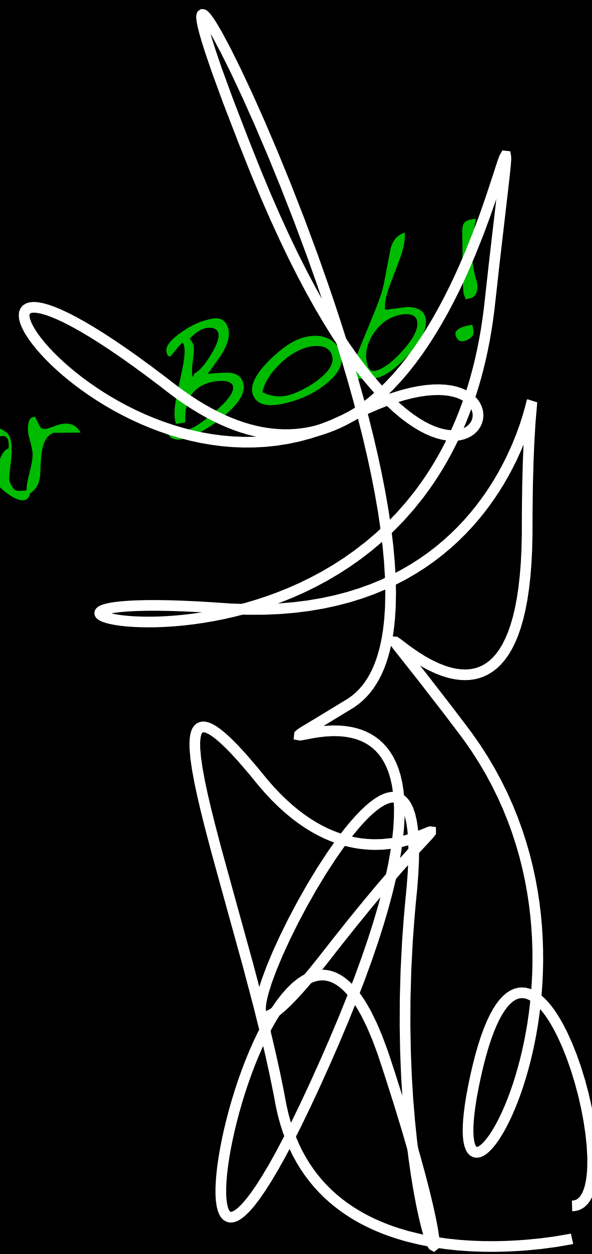


(noise)

1. Communication

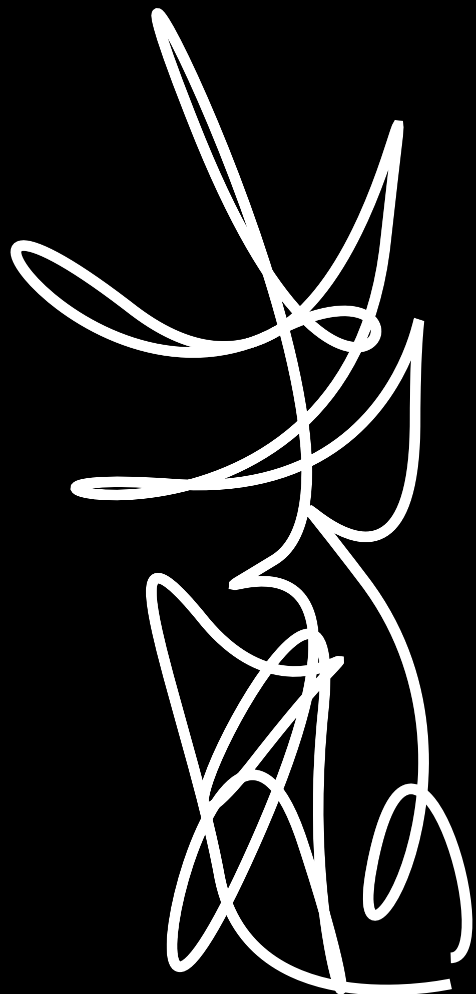
Deep throat?

Dear Bob!



(noise)

1. Channels & capacity

Noise  modelled as "channel":

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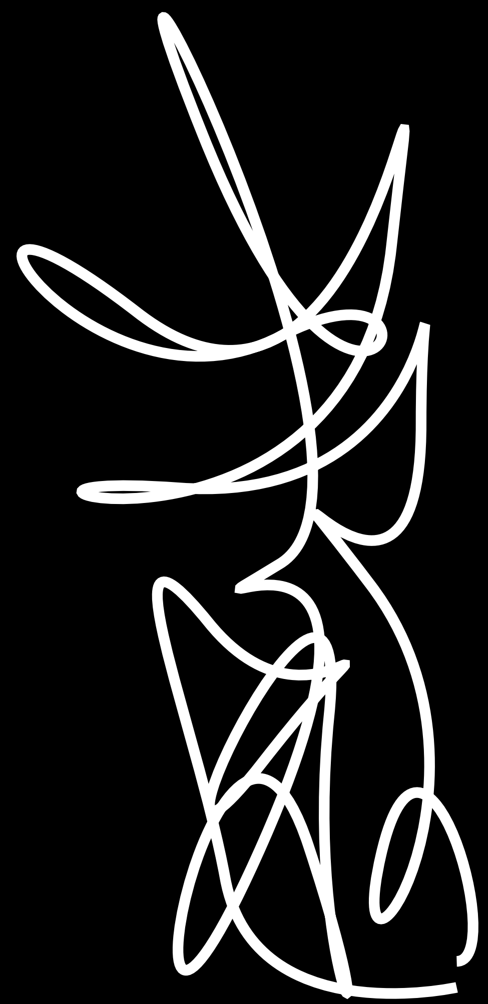
Noise  modelled as "channel":

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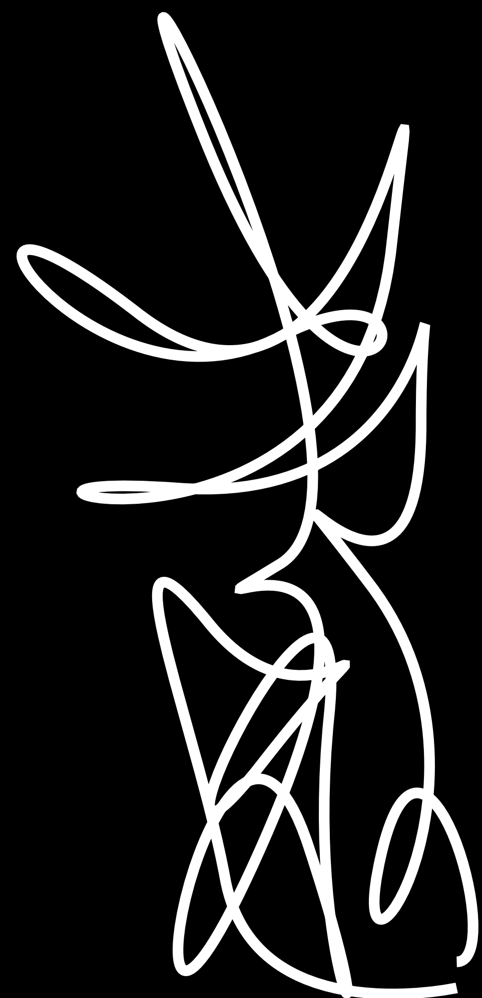
Noise  modelled as "channel":

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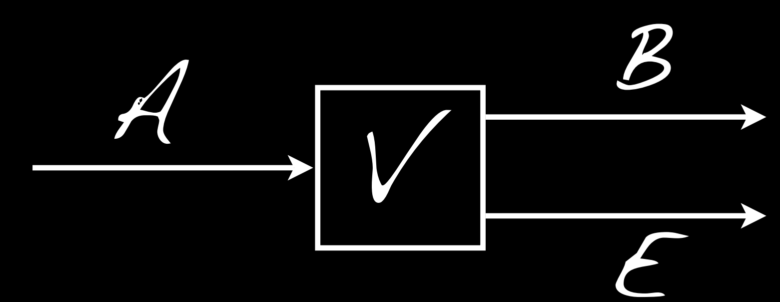

Complementary channel:

$$\widehat{N}(\rho) = Tr_B \, V \rho V^\dagger.$$

1. Channels & capacity

Ex: 1) Noiseless channel = identity id_A.

2) Constant channel $K(\rho) = \omega_0$.

3) Depolarizing channels

4) Amplitude damping channels

5) Phase damping channels

6) Erasure channel $\mathcal{E}_q(\rho) = (1-q)\rho \oplus q|*\rangle\langle *|$

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(Later in this talk, we'll look at some special classes: degradable, Hadamard, entanglement-breaking, ...)

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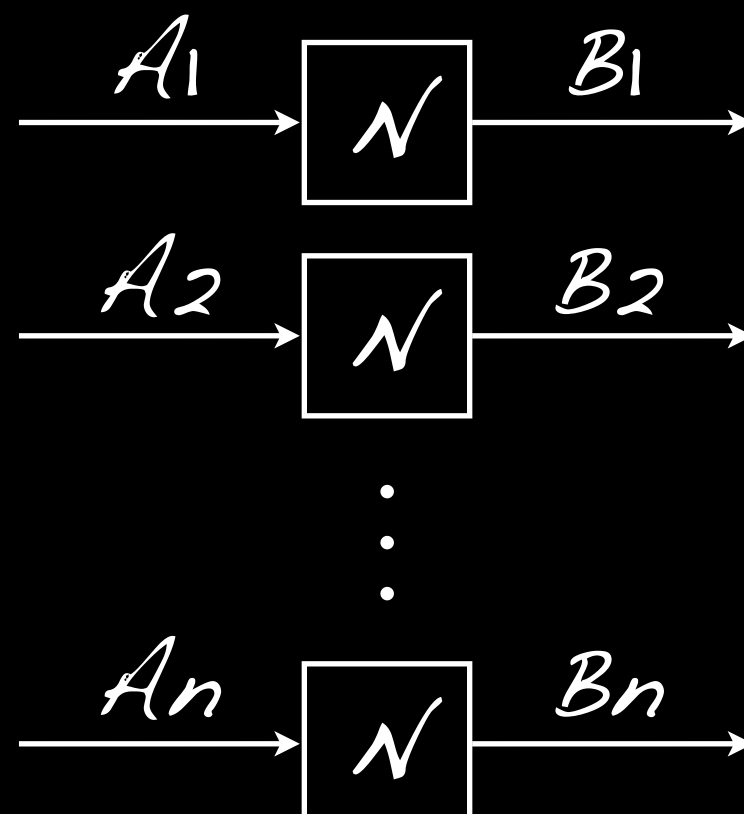

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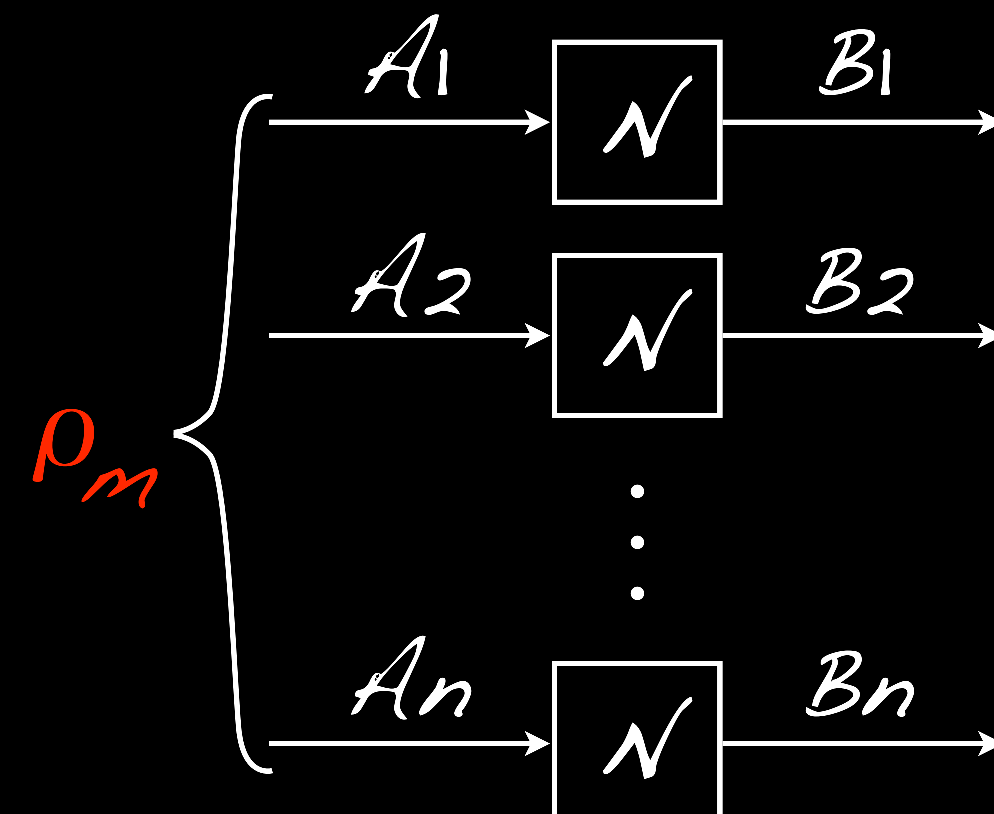
m



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$k=k(n,\varepsilon)$ bits

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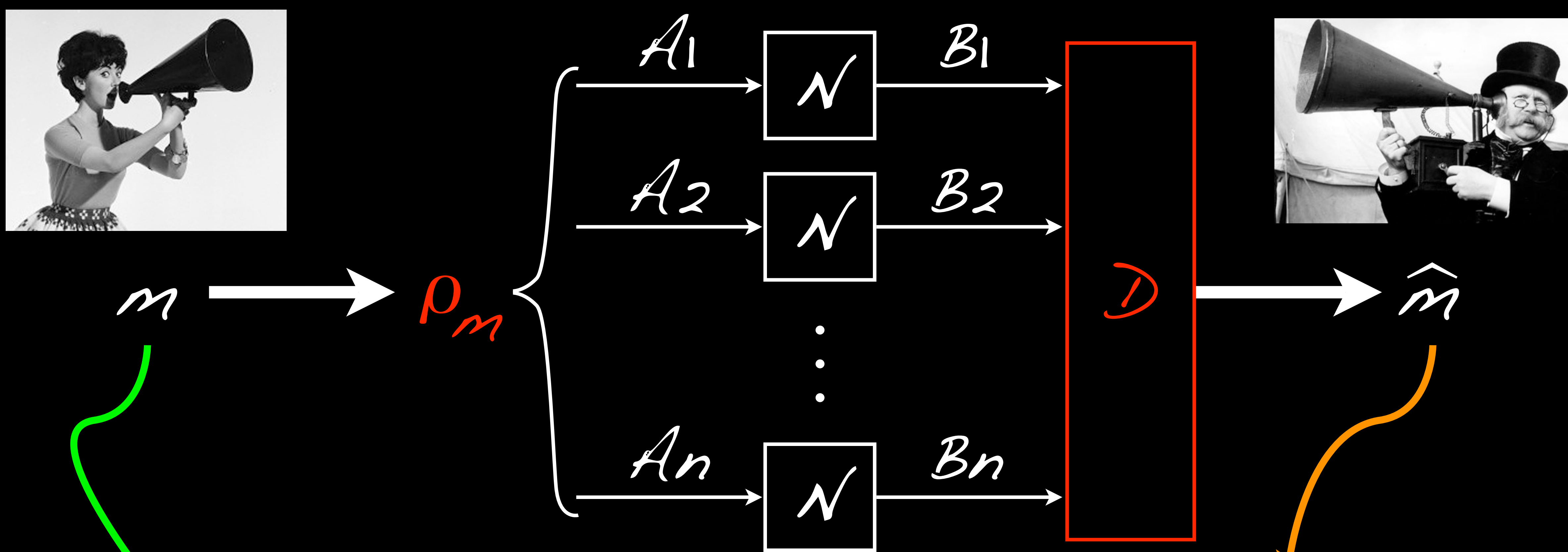

message of $k = k(n, \varepsilon)$ bits

$= m$ with prob $\geq 1 - \varepsilon$

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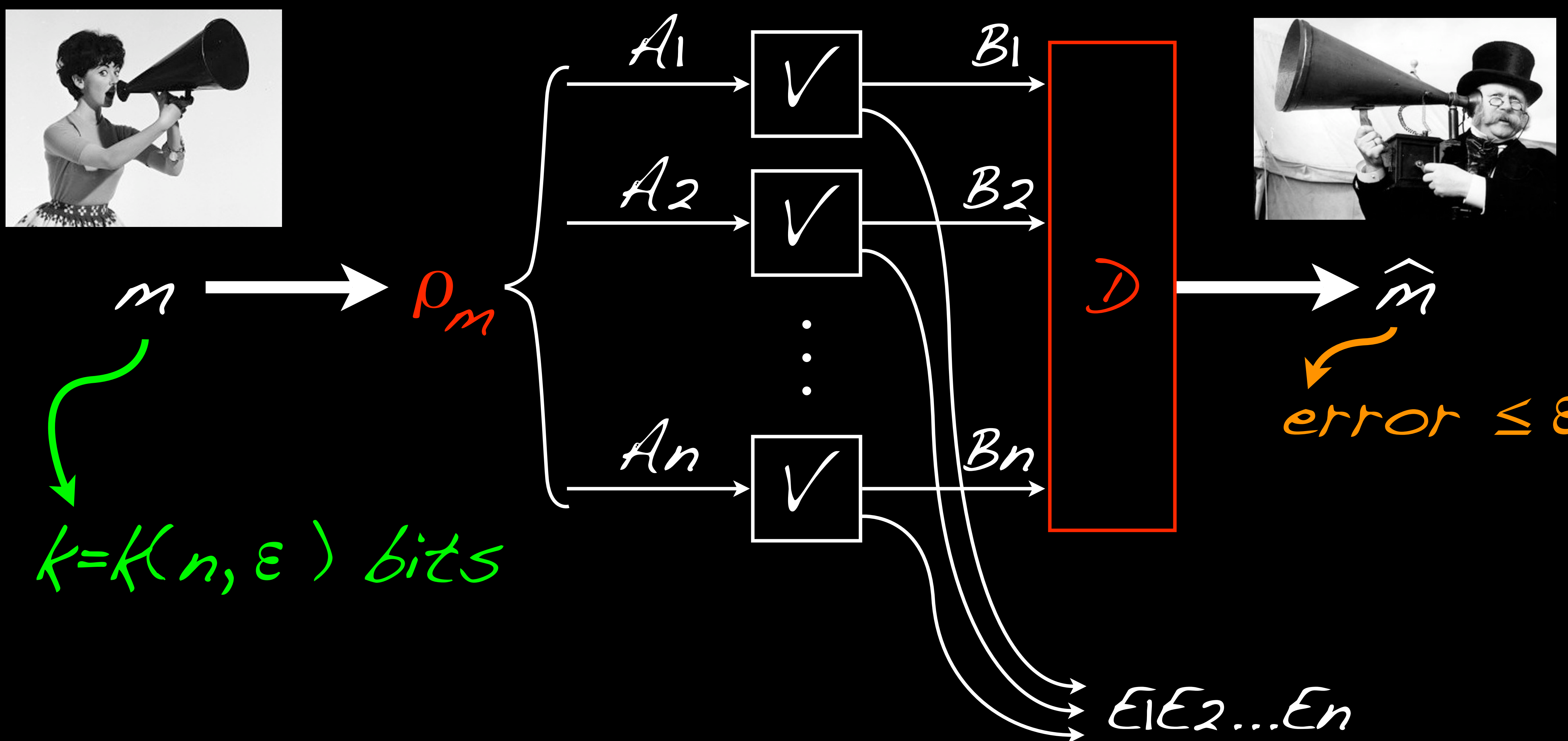
Quantum capacity $Q(\mathcal{N}) :=$ maximum qubit rate for asymptotically faithful transmission.

…and a veritable "zoo" when allowing other free resources: $\mathcal{E}, \leftarrow, \rightarrow, \leftrightarrow, \ldots$

Private capacity $P(N)$:= maximum cbit rate $\frac{k}{n}$ for asymptotically error-free and secret transmission over $N^{\otimes n}$.



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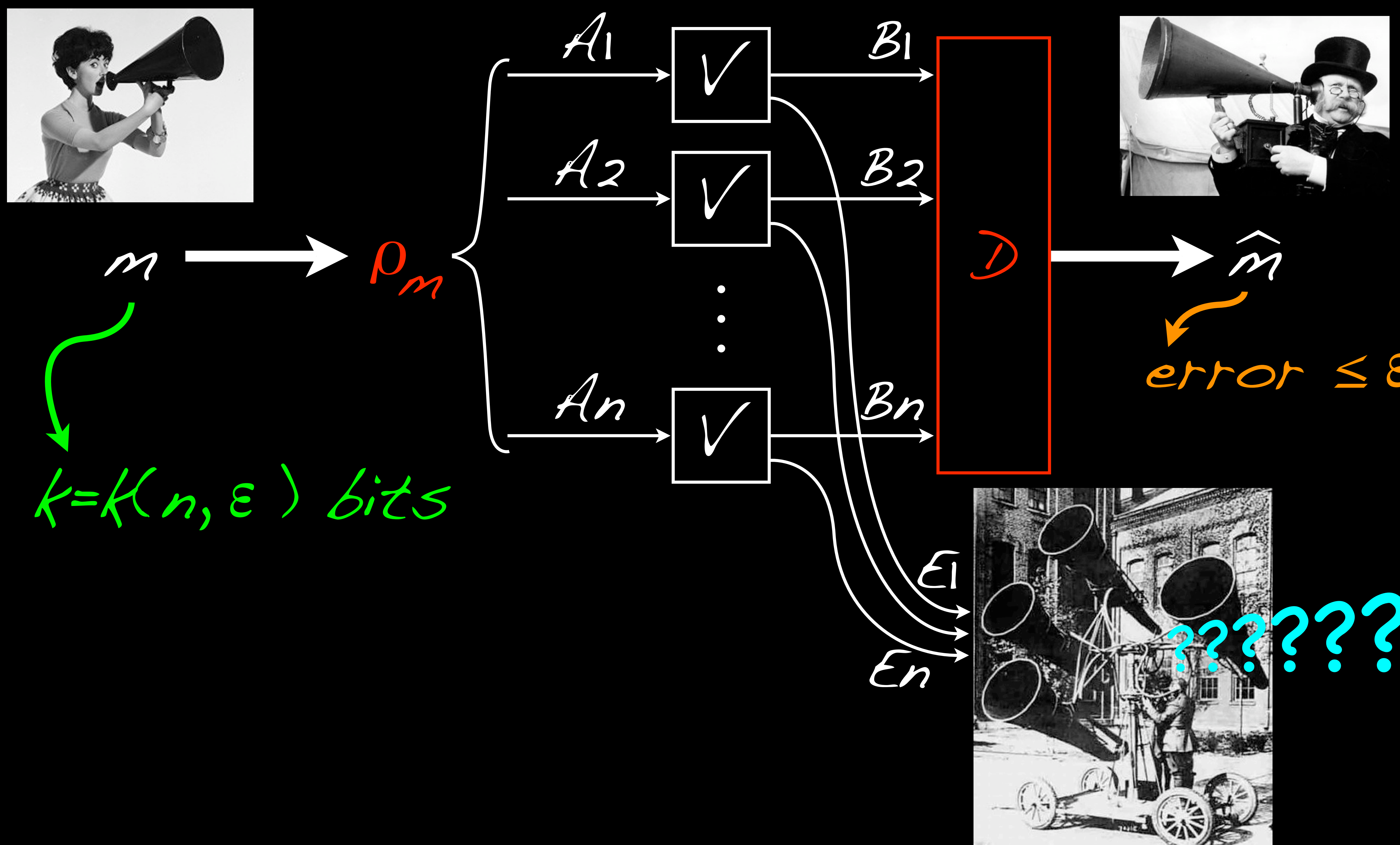

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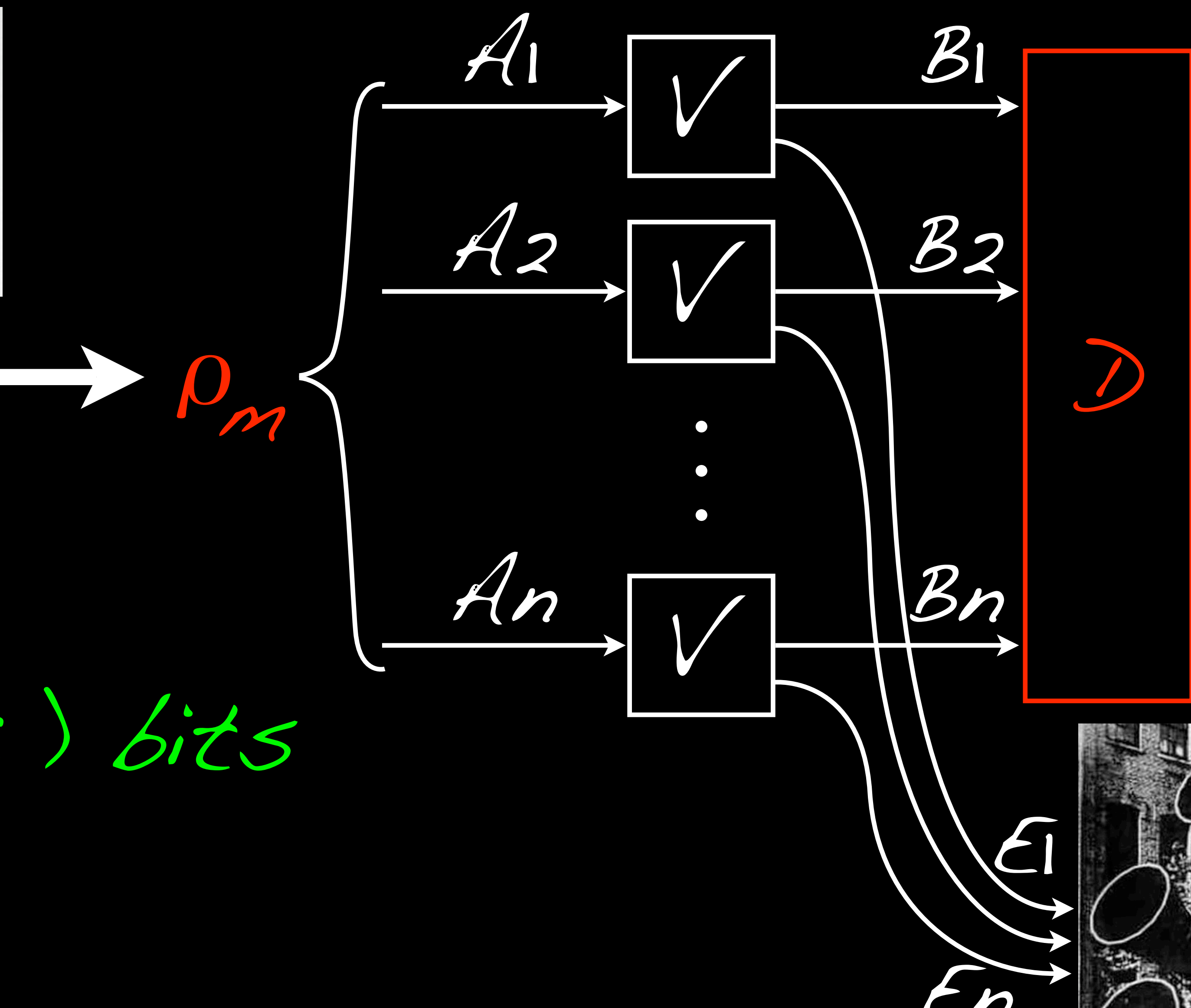
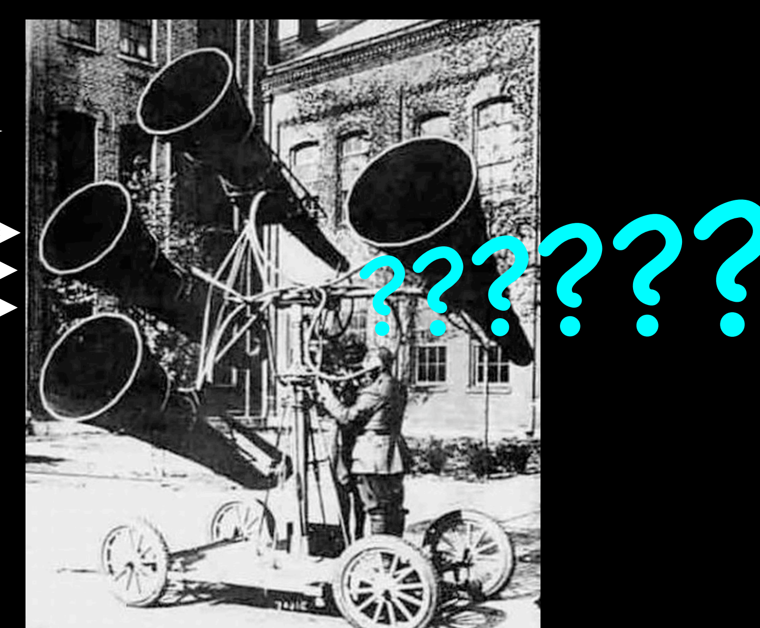


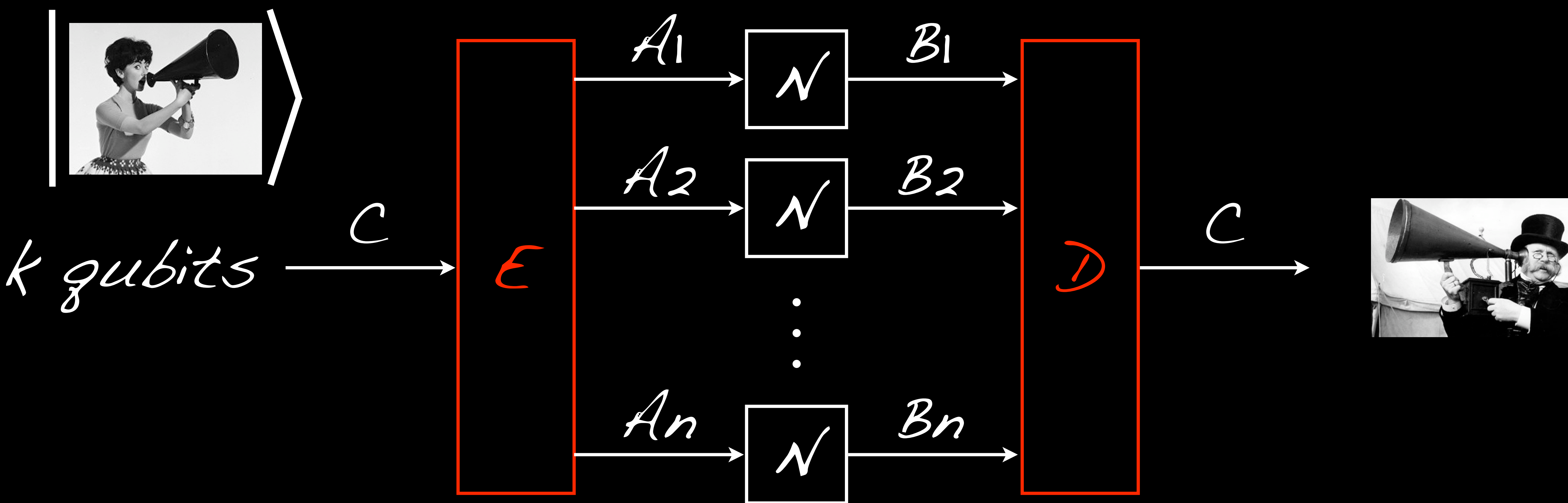
$k = k(n, \varepsilon)$ bits

error $\leq \varepsilon$



Secret: $\| \hat{\mathcal{N}}^{\otimes n}(\rho_m) - \omega_0 \|_1 \leq \delta$

Quantum capacity $Q(\mathcal{N})$ requires en- and decoding by cptp maps $\mathcal{E}$, $\mathcal{D}$:



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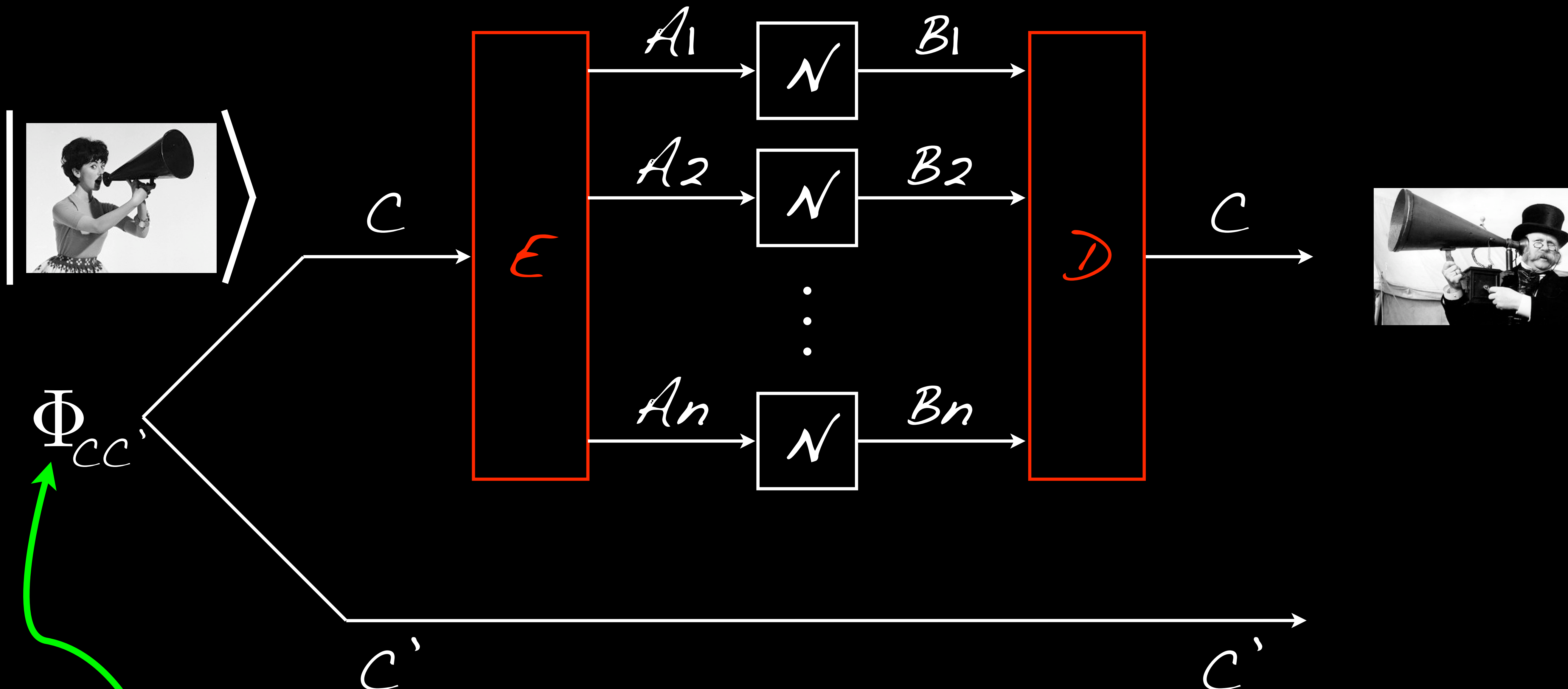

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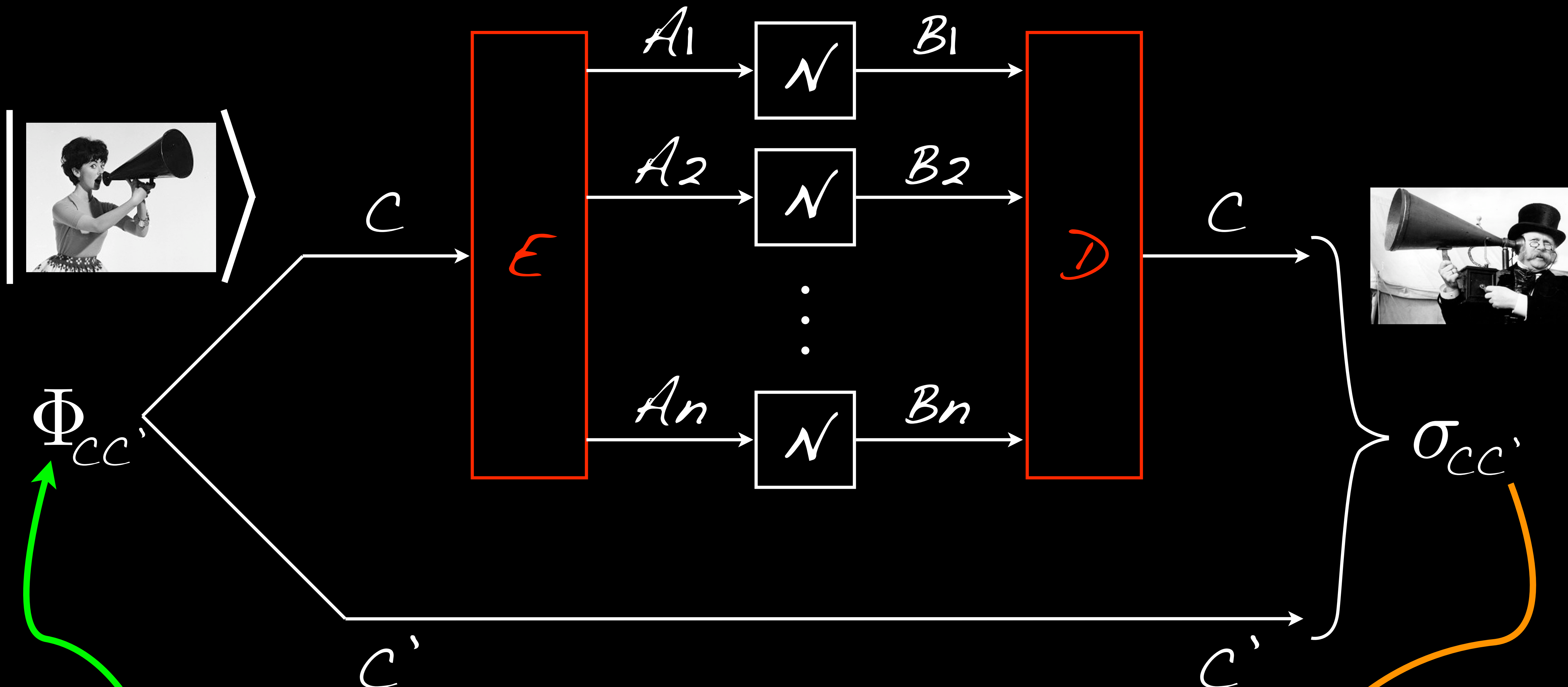

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Approximates input:
$P(\Phi, \sigma) \leq \varepsilon$.

Digression on fidelity:

$$F(\rho,\sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_1$$

$$= \max |\langle\psi|\varphi\rangle| \text{ s.t.}$$

$|\psi\rangle$ purifies ρ, $|\varphi\rangle$ purifies σ.

$P(\rho,\sigma) := \sqrt{1-F(\rho,\sigma)^2}$ is a metric on states;

...and so is $A(\rho,\sigma) := \arcsin P(\rho,\sigma)$.

Note: Both are equivalent to the trace distance $\|\rho-\sigma\|_1$.

[cf. M. Tomamichel, PhD thesis, arXiv:1203.2142]

Outline

1. Quantum channels and their capacities ✓
2. Entropic capacity formulas; weak converse
3. What is a strong converse?
4. Ideal channel (warm-up); simulation argument
5. Rényi divergence paradigm: classical capacity
6. Min-entropies: "pretty strong" converse
7. End credits

2. Capacity formulas and weak converse

Thm (Holevo and Schumacher/Westmoreland, 1973 and 1996/7):

$$C(N) = \lim_{n\to\infty} \frac{1}{n} \chi(N^{\otimes n}), \text{ with}$$

$\chi(N) = \max I(X:B)$ wrt. $\{p_x, \rho_x\}$ and

$$\rho_{XB} = \sum_x p_x |x\rangle\langle x| \otimes N(\rho_x).$$

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Holevo information $S(\rho_B) - \sum_x p_x S(\mathcal{N}(\rho_x))$

Von Neumann entropy: $S(\rho) = -\mathrm{Tr}\, \rho \log \rho$

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However, for some classes of channels it is, and we know the classical capacity $C(\mathcal{N})$ as $\chi(\mathcal{N})$.

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Uses only strong subadditivity (SSA) and continuity of von Neumann entropy (Fannes inequality): $S(\rho) = -\mathrm{Tr}\,\rho \log \rho$.

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...is the implied tradeoff real?

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coherent information

$$Q^{(1)}(N) = \max I(A\rangle B)$$

$$= \max S(N(\rho)) - S(\hat{N}(\rho)) \text{ wrt. } \rho$$

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(Btw: also additivity issue with these!)

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...so what about rates above capacity?

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By contrapositive: If error < 1, then asymptotically the rate $\frac{k}{n}$ is bounded by the capacity.

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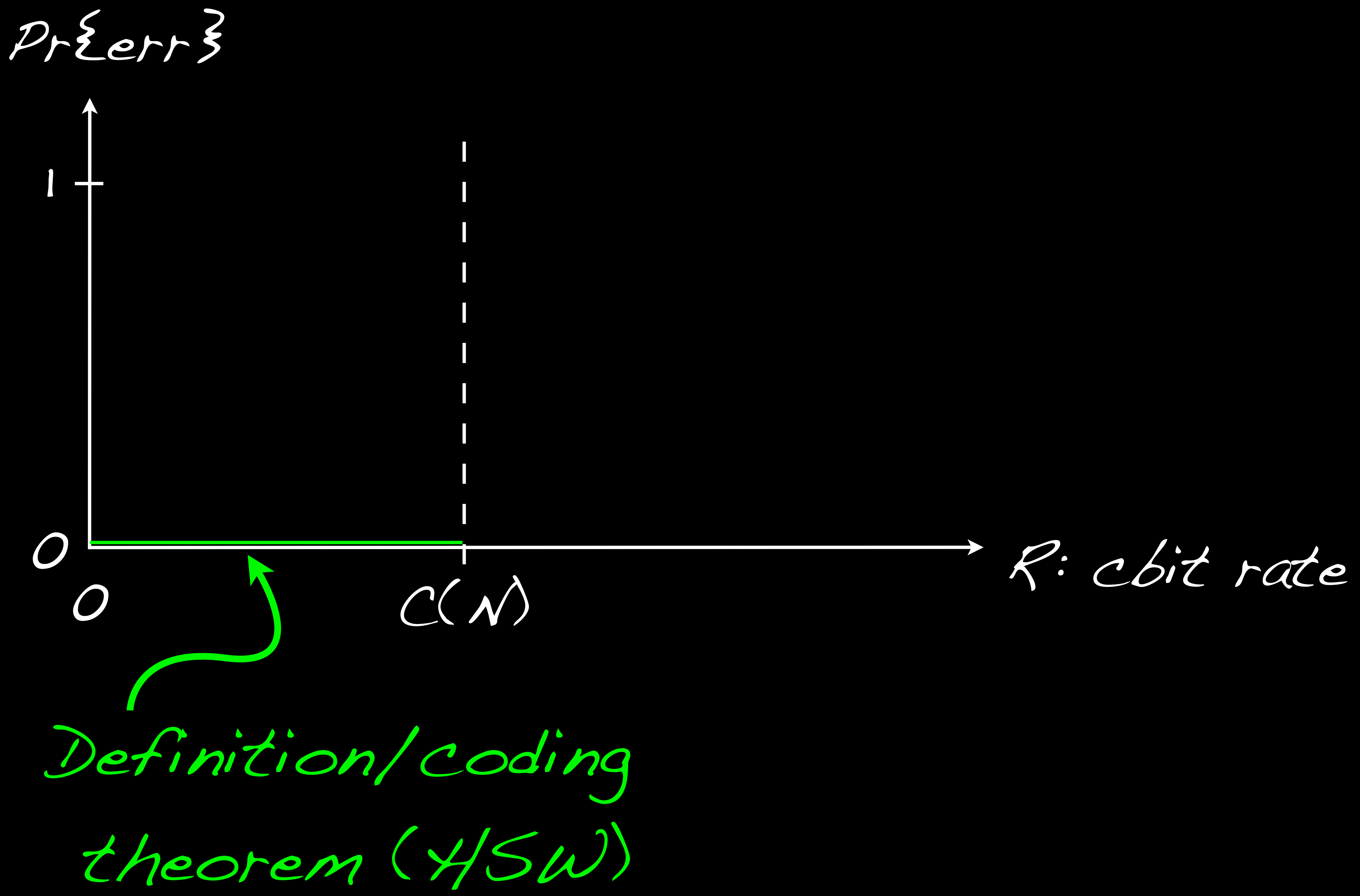
- Classical capacity of covariant channels [Koenig/Wehner, PRL 103:070504 (2009)]

Rate vs asymptotic error:



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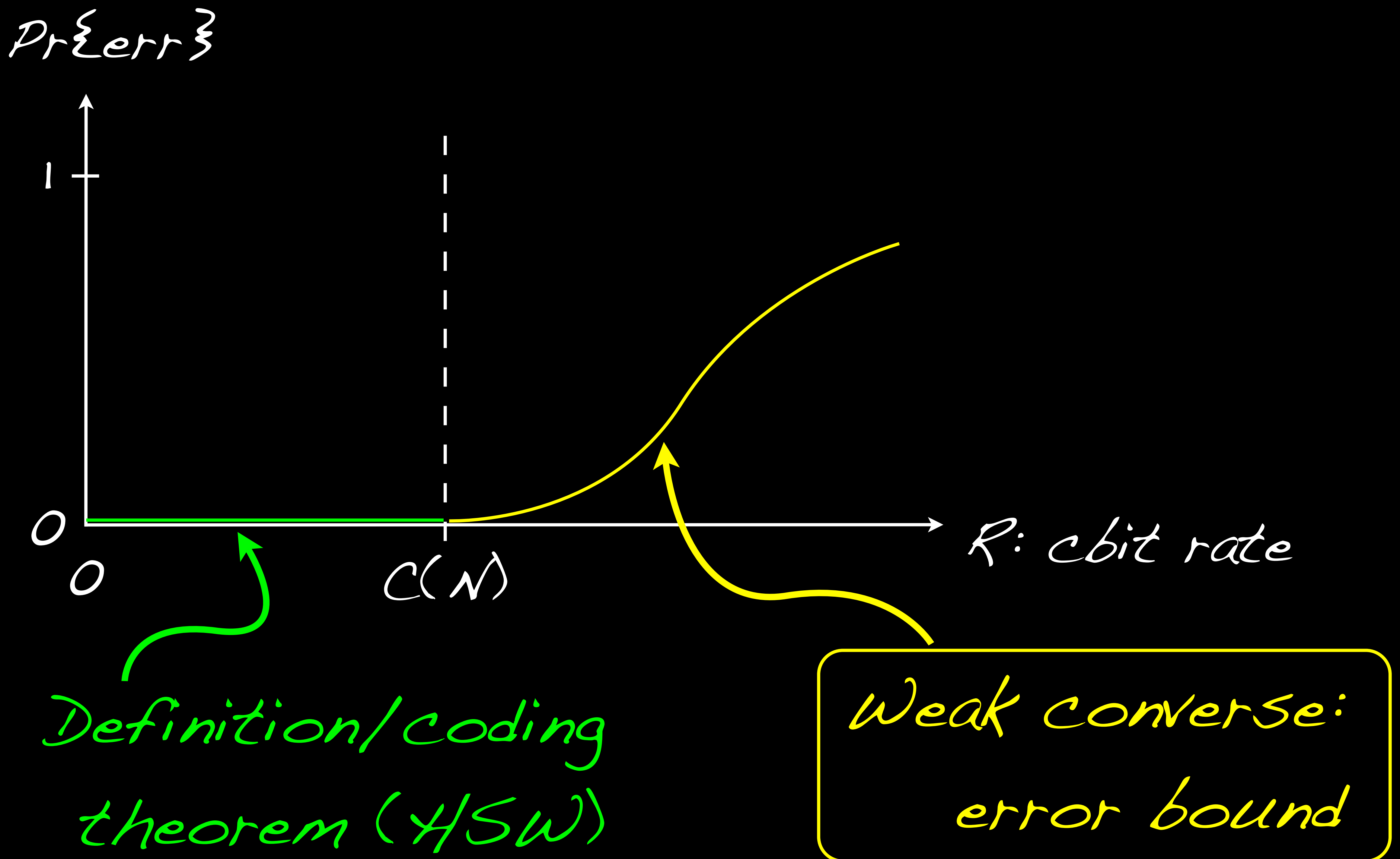

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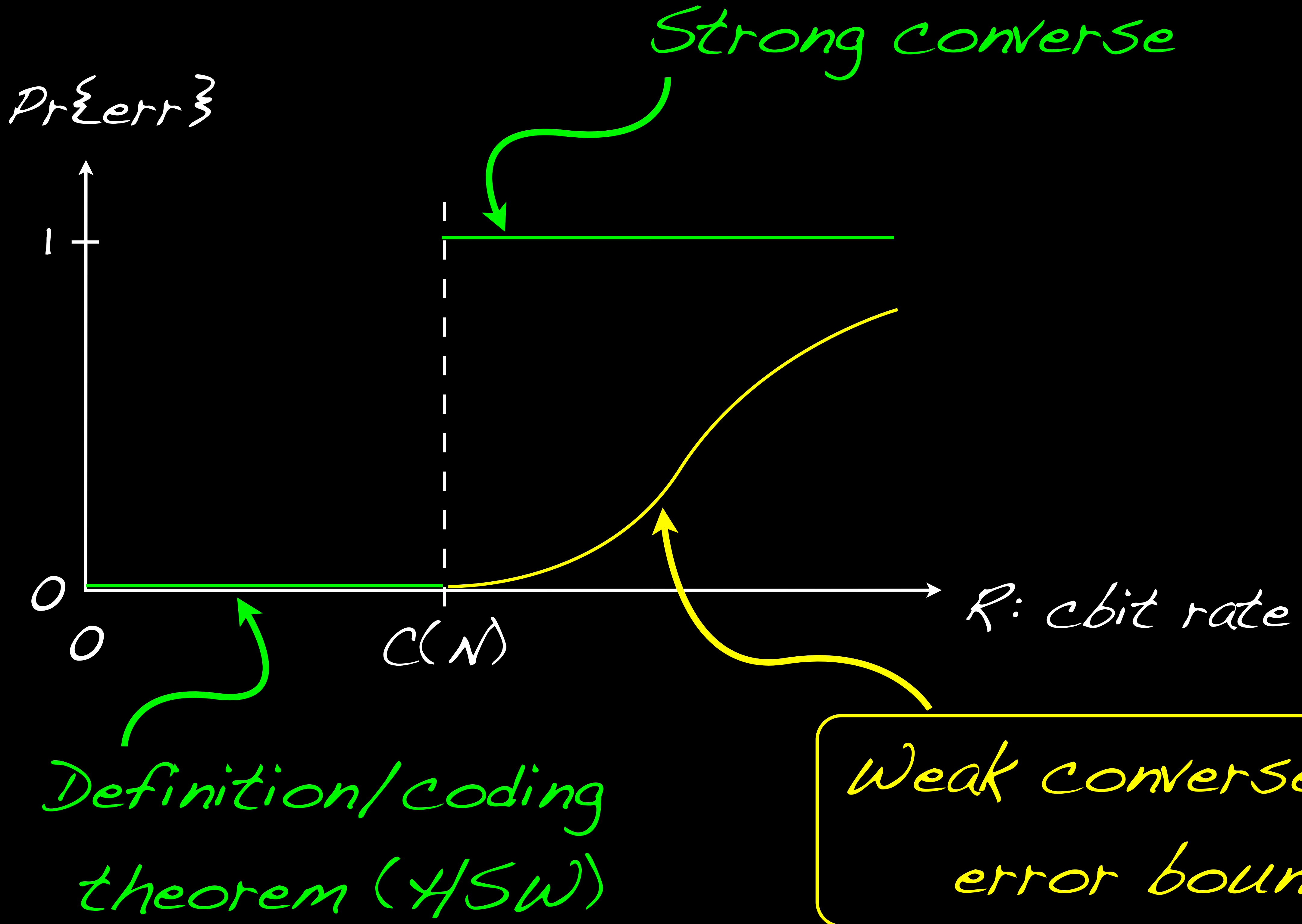

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Hence $Q(N) \leq P(N) \leq C(N)$ in general. Since $Q(\mathrm{id}_2) = P(\mathrm{id}_2) = C(\mathrm{id}_2) = 1$, enough to show it for the classical capacity.

Warm-up: strong converse for the noiseless qubit channel id_2.

Encode M message into id_L via states ρ_m and POVM elements D_m to decode:

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For n uses of the channel and rate $R>1$: $L=2^n$ and $M=2^{nR}$, so $\varepsilon \geq 1-2^{-n(R-1)}$. QED

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The simulation argument: If you can simulate a channel N by id_2 at rate K, then $C(N) \leq K$ and for rates $R > K$, the error $\varepsilon \geq 1 - 2^{-n(R-K)}$.

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In particular: If $K = C(N)$, strong converse holds. Almost only trivial cases, except:

Thm (Wilde/AW, 1308.6732): For pure loss optical channel w/ transmissivity η and maximum mean photon number P, $C = g(\eta P)$, and the strong converse holds.

The simulation argument: If you can simulate a channel N by id_2 at rate K, then $C(N) \leq K$ and for rates $R > K$, the error $\varepsilon \geq 1 - 2^{-n(R-K)}$.

More interesting with free resources, eg.

$C_E(N)$ = ent.-assisted classical capacity

= minimal simulation cost assisted by ent. ("Qu. Reverse Shannon Thm.")

Ie. strong converse holds for C_E.

[Bennett et al., IEEE-IT 48:2637 (2002); Bennett et al. 0912.5537]

[Cf. Berta et al., IEEE-IT 59:6770 (2013) - $P(N)$ bound]

5. Rényi divergences for C

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What can we do for $C(N)$? Nothing general it seems... However, unifying and extending the earlier results of Ogawa/Nagaoka, AW and König/Wehner:

Thm (Wilde/AW/Yang, 1306.1586): If N is entanglement-breaking (EB) or Hadamard (H), then for any code w rate $R > C(N)$, Pr{err} converges to 1, exponentially fast in the number n of channel uses.

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In other words, these channels satisfy the strong converse.

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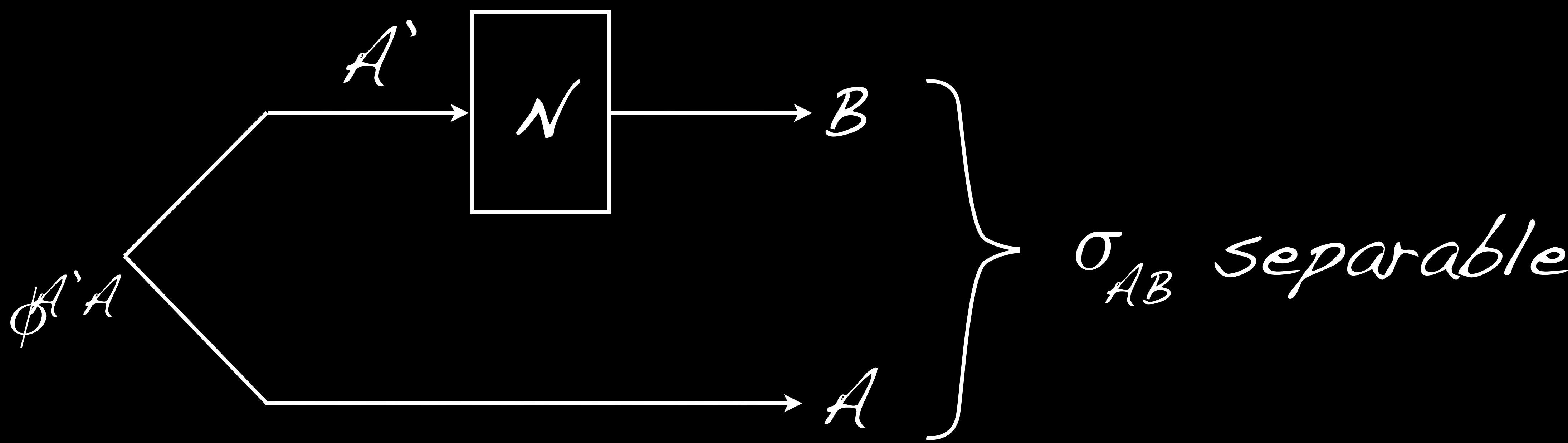
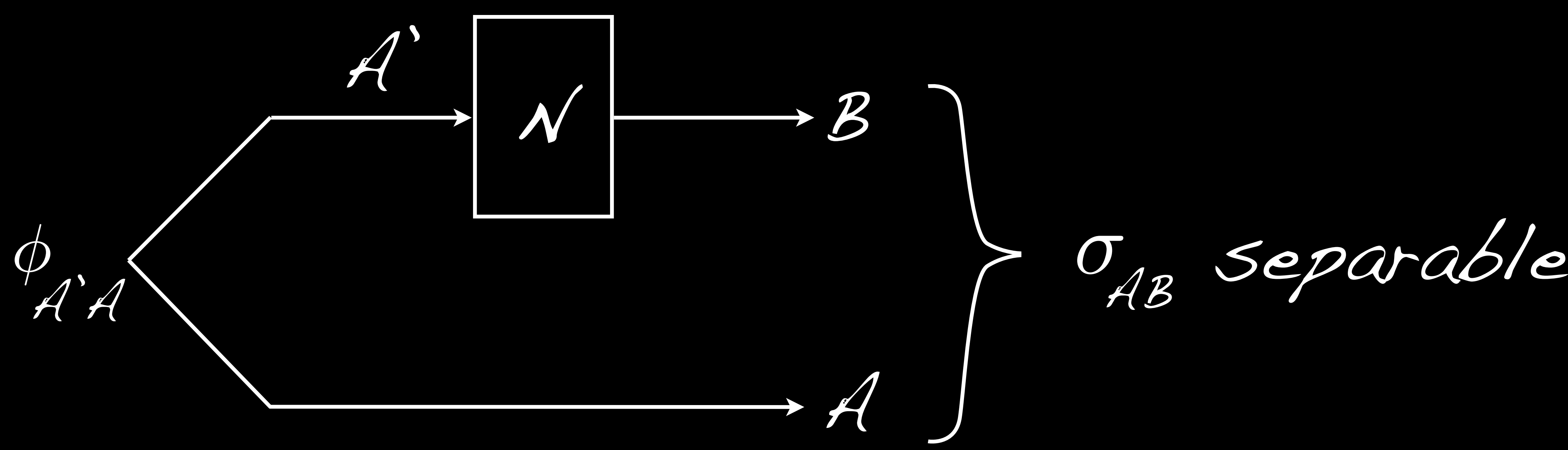
Entanglement-breaking (EB) channels:



Complementary to these:

Hadamard channels ($\mathcal{H}$)

Entanglement-breaking (EB) channels:



Fact: $\mathcal{N}$ entanglement-breaking iff

$$\mathcal{N}(\rho) = \sum_i Tr(\rho M_i)\sigma_i \quad s.t. \quad \sum_i M_i = \mathbb{1}$$

$$= \sum_j |\beta_j\rangle\langle\alpha_j|\rho|\alpha_j\rangle\langle\beta_j|$$

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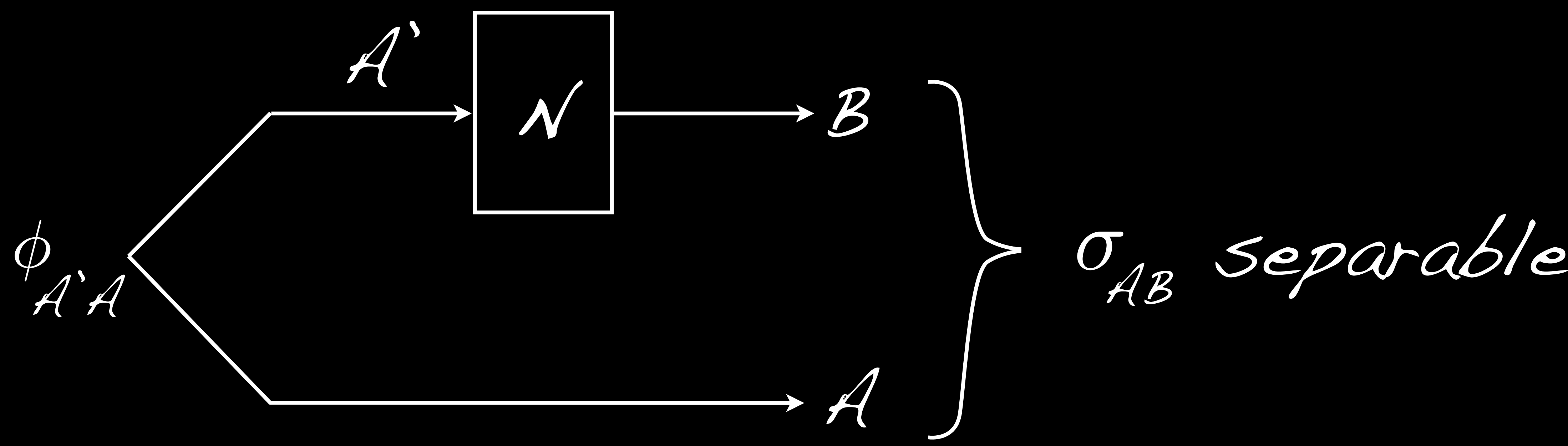

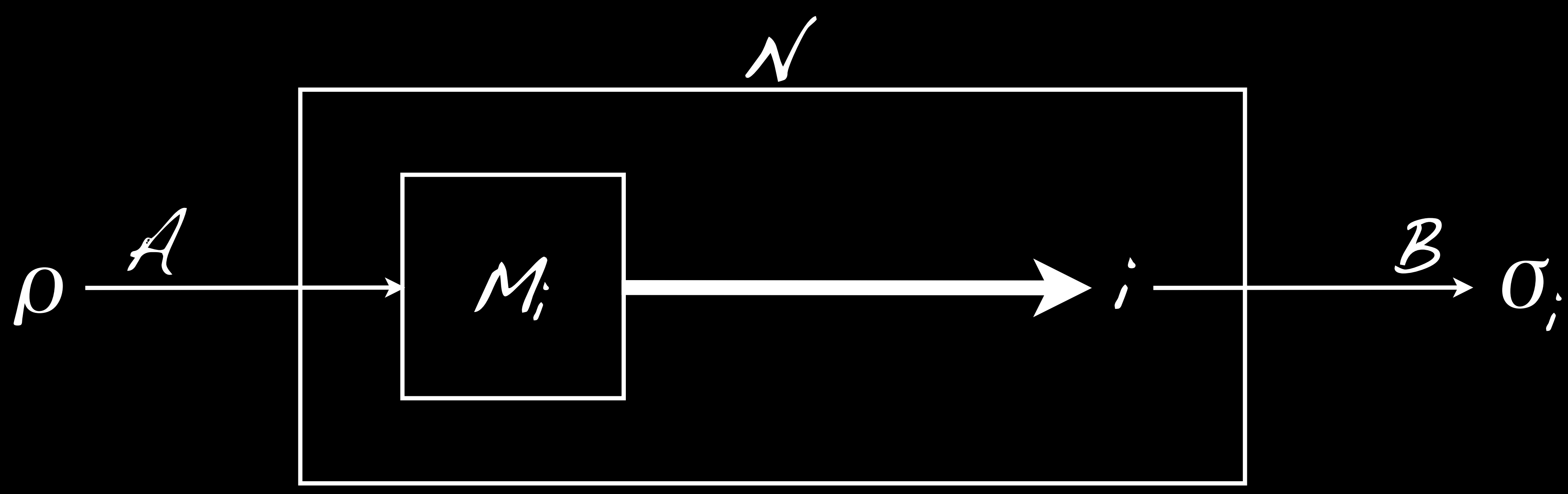

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Schur product

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Channels of this form: Hadamard channels

Examples – Entanglement-breaking channels:

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4) Cloning channels [cf. Brádler, IEEE-IT 2011]

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Fundamental property is monotonicity:

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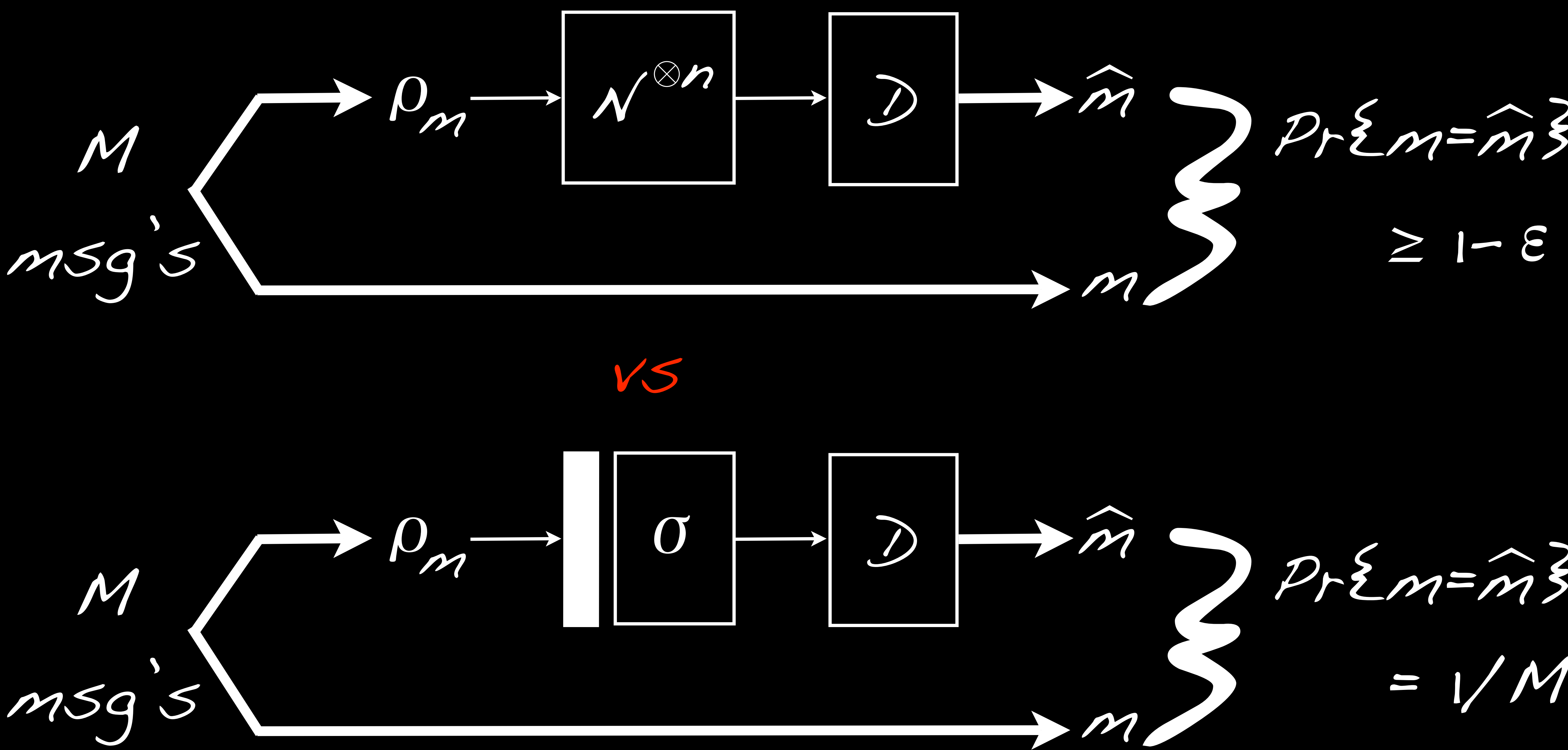

[Cf. Polyanskyi/Verdú, Proc. 48th Allerton CCC, 2010]

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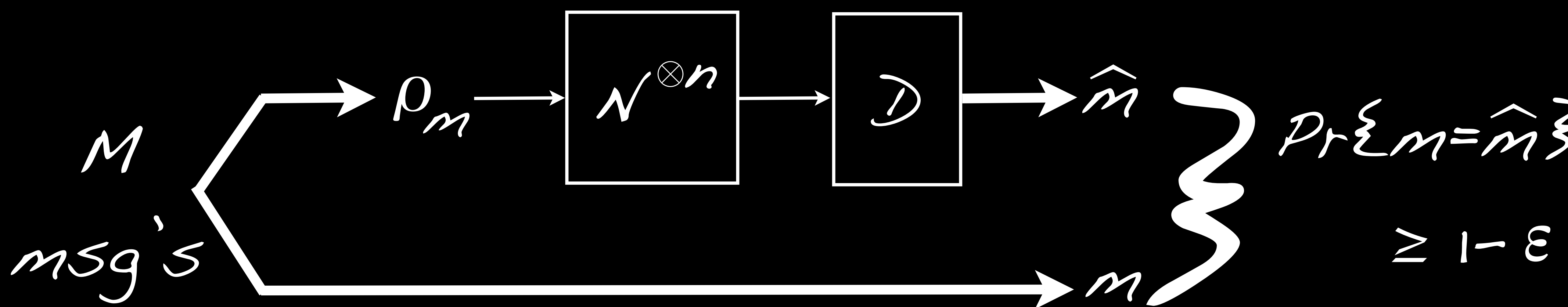

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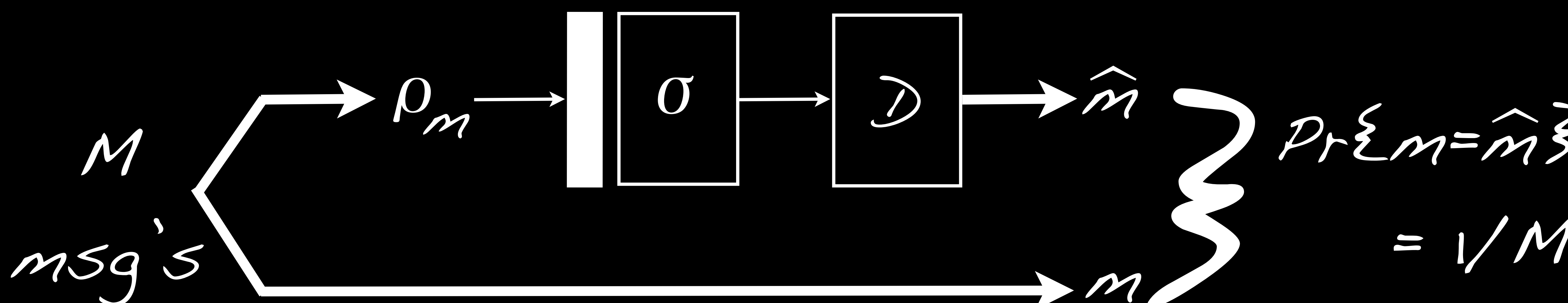

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(For usual relative entropy, we recover the previous weak converse.)

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L.h.s.: $\frac{\alpha}{\alpha - 1} \log(1-\varepsilon) + \log M.$

For $\log M = nR$:

$$1-\varepsilon \leq \exp\left\{-n\,\frac{\alpha-1}{\alpha}\left(R-\tilde{\chi}_\alpha(N)\right)\right\},$$

which is exponentially small for $R > \chi(N)$ and $\alpha > 1$ small enough.

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Combining with simulation argument and recent additivity of minimum output (Rényi) entropies [Giovannetti/García-Patrón/Cerf/Holevo, 1312.6225]: Strong converse for covariant Gaussian channels. ✓

[Bardhan/García-Patrón/Wilde/AW, 1401.4161]

6. Min-entropies: "pretty strong" converse for Q

Stinespring: $\mathcal{N}(\rho) = \mathrm{Tr}_E\, V\rho V^\dagger,$

with an isometry $V: A \hookrightarrow B \otimes E$.

Complementary channel:

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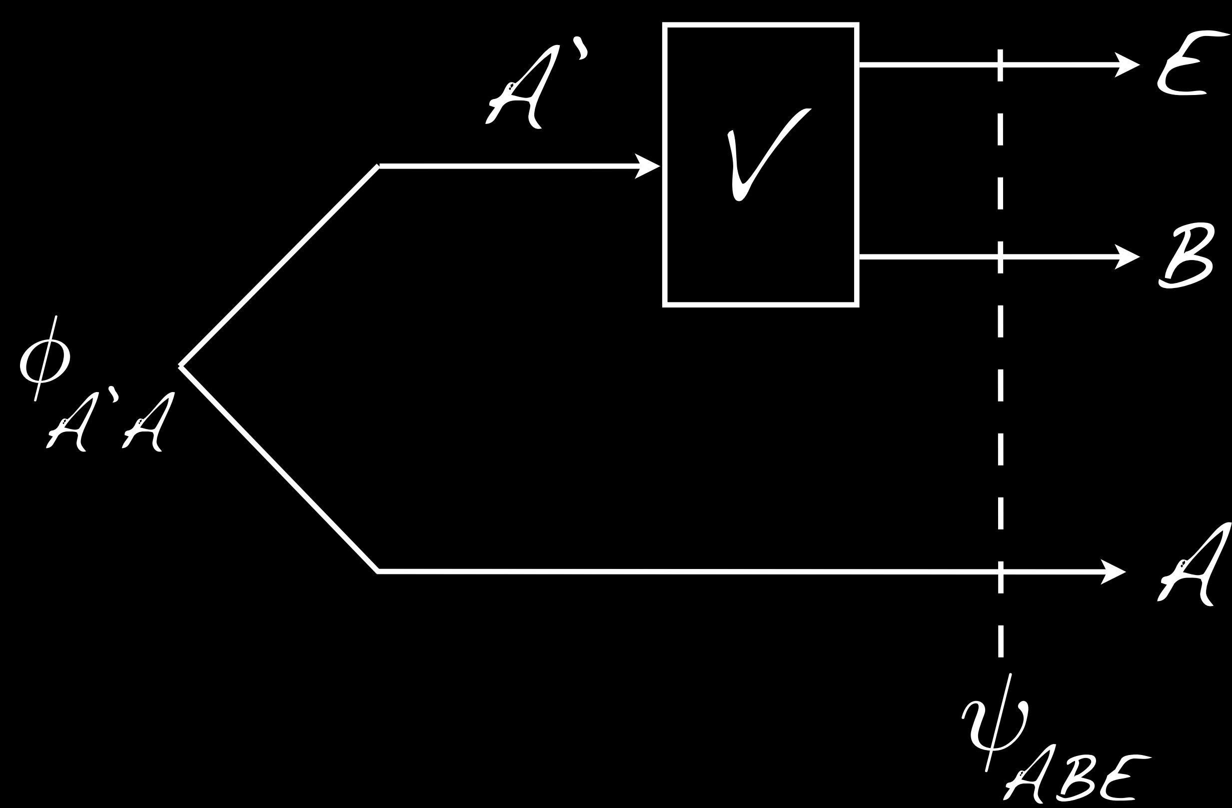
$\mathcal{N}$ is degradable if there exists a cptp map $\mathcal{D}$ s.t. $\widehat{\mathcal{N}} = \mathcal{D} \circ \mathcal{N}$. Vice-versa: anti-degradable.

Degradability in the Church of the Larger Hilbert Space:



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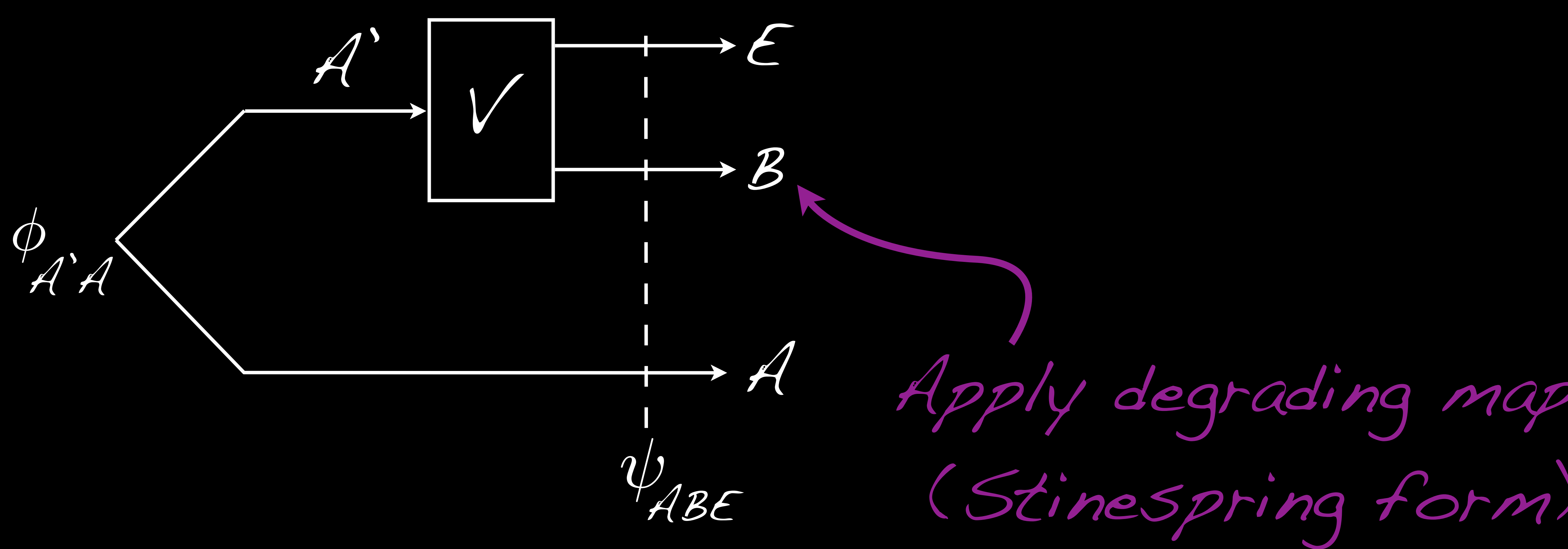

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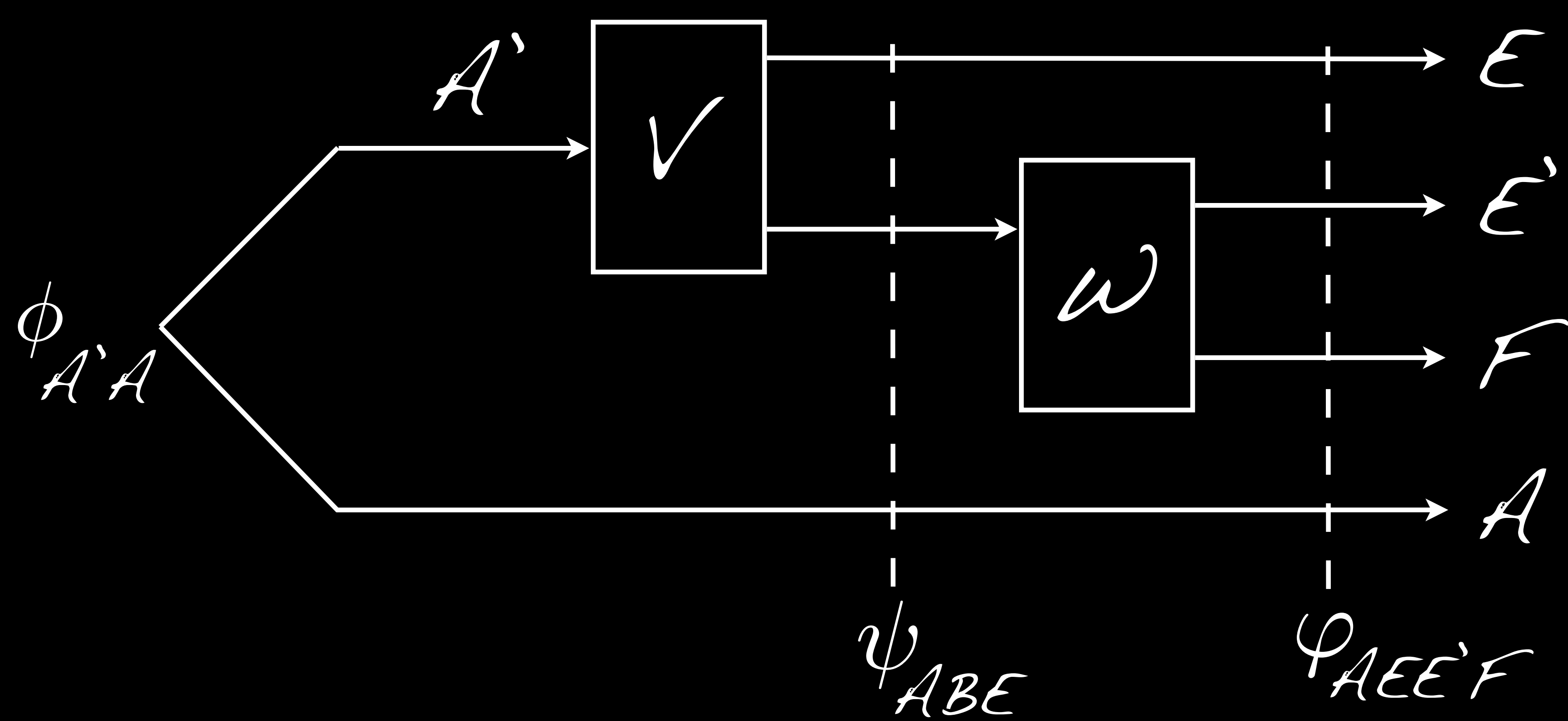

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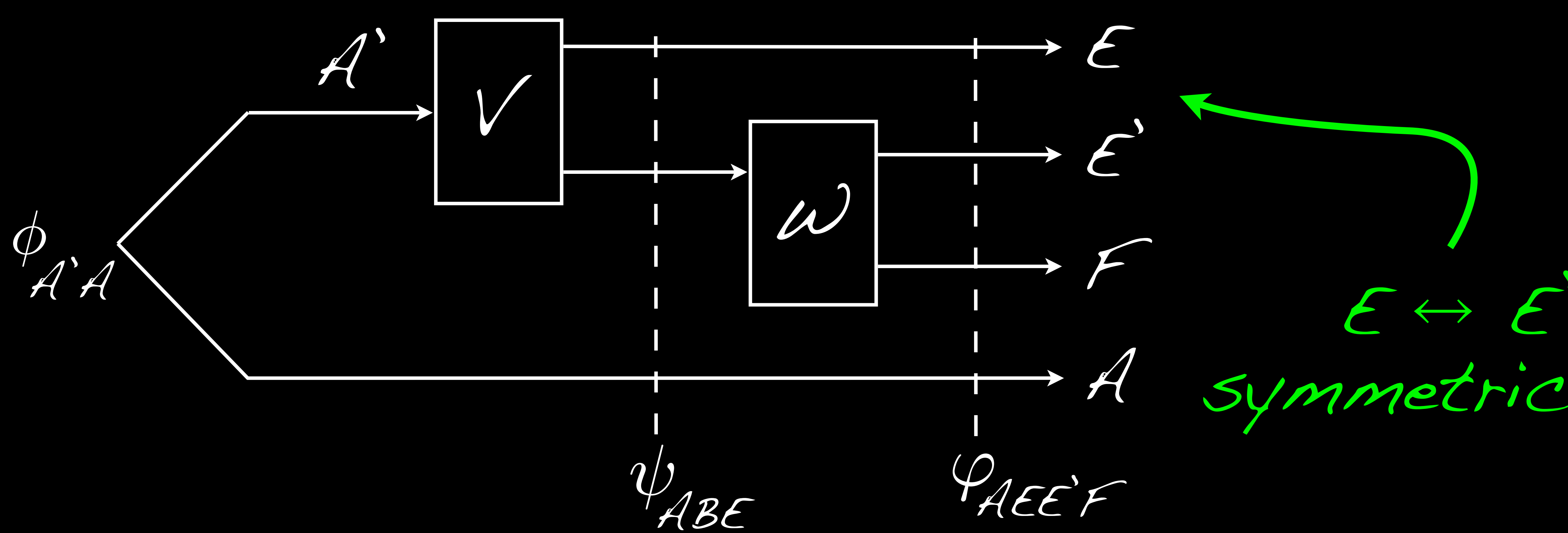

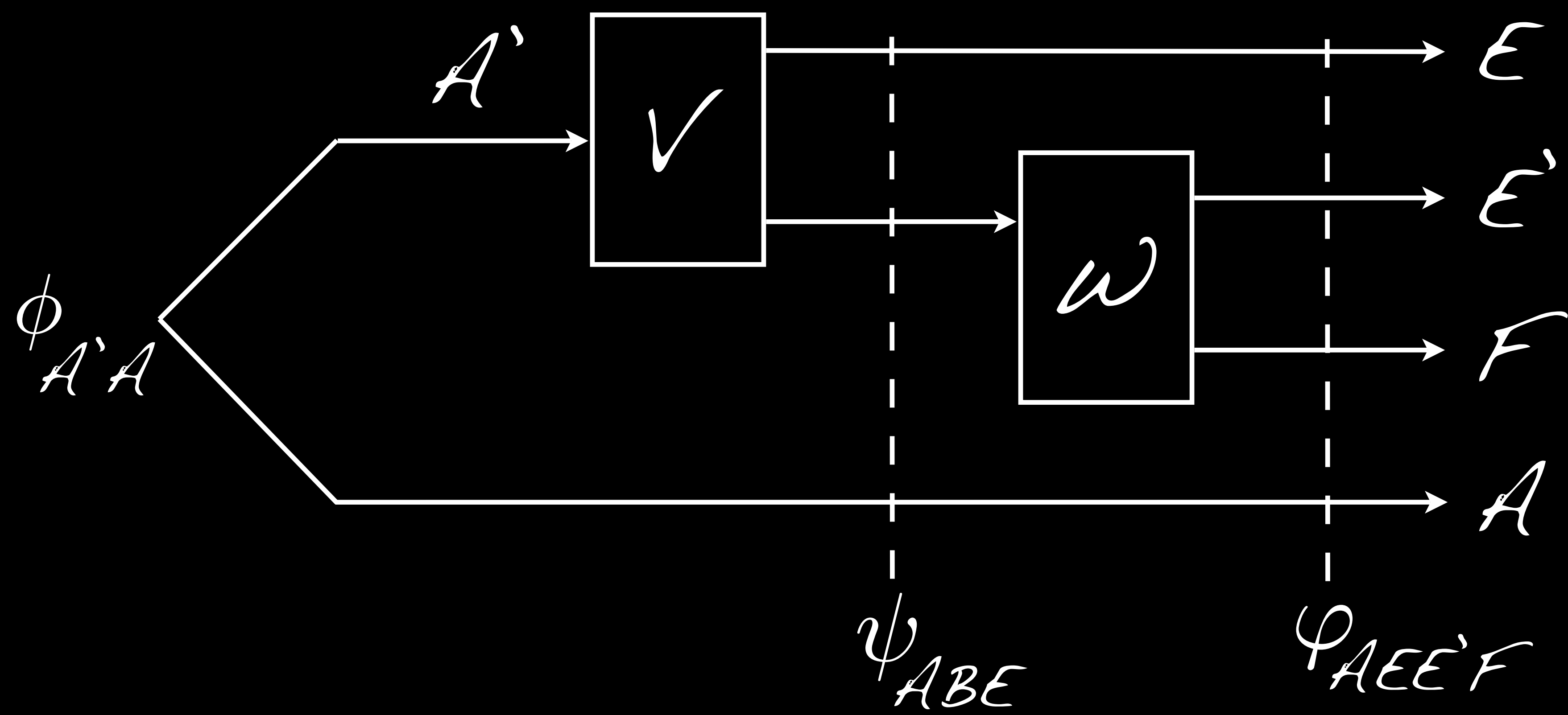

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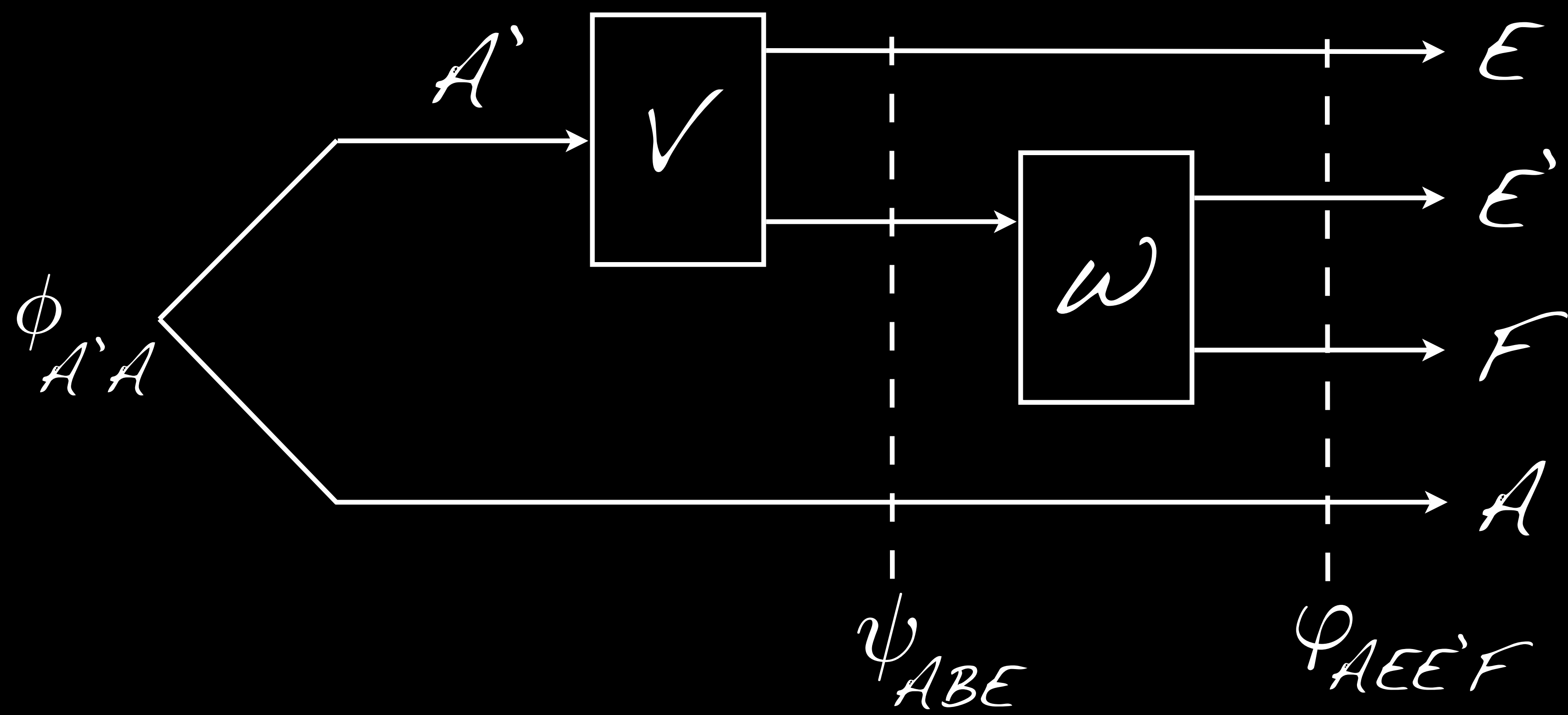

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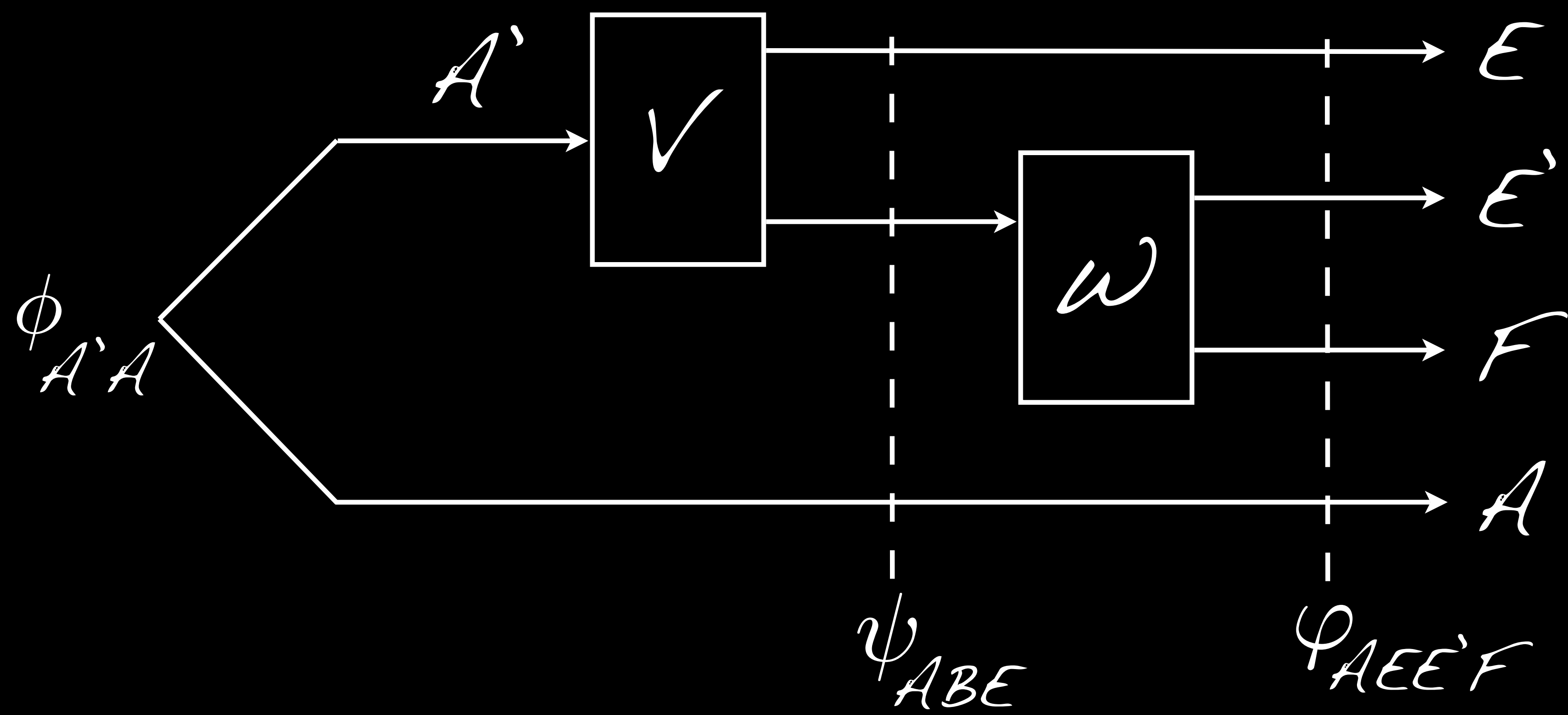

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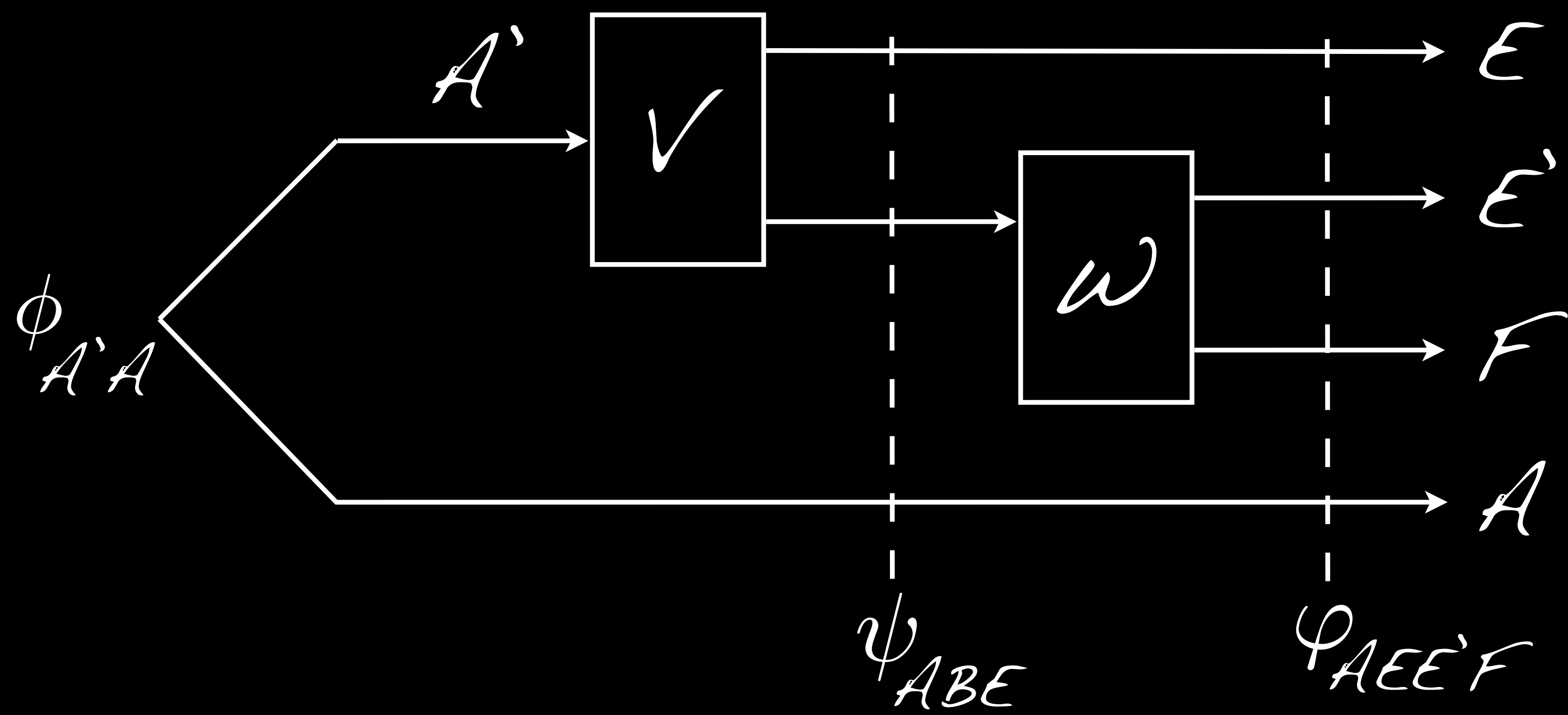
2) Amplitude damping channel
3) Symmetric channels, i.e. trivial F, for instance 50% erasure channel

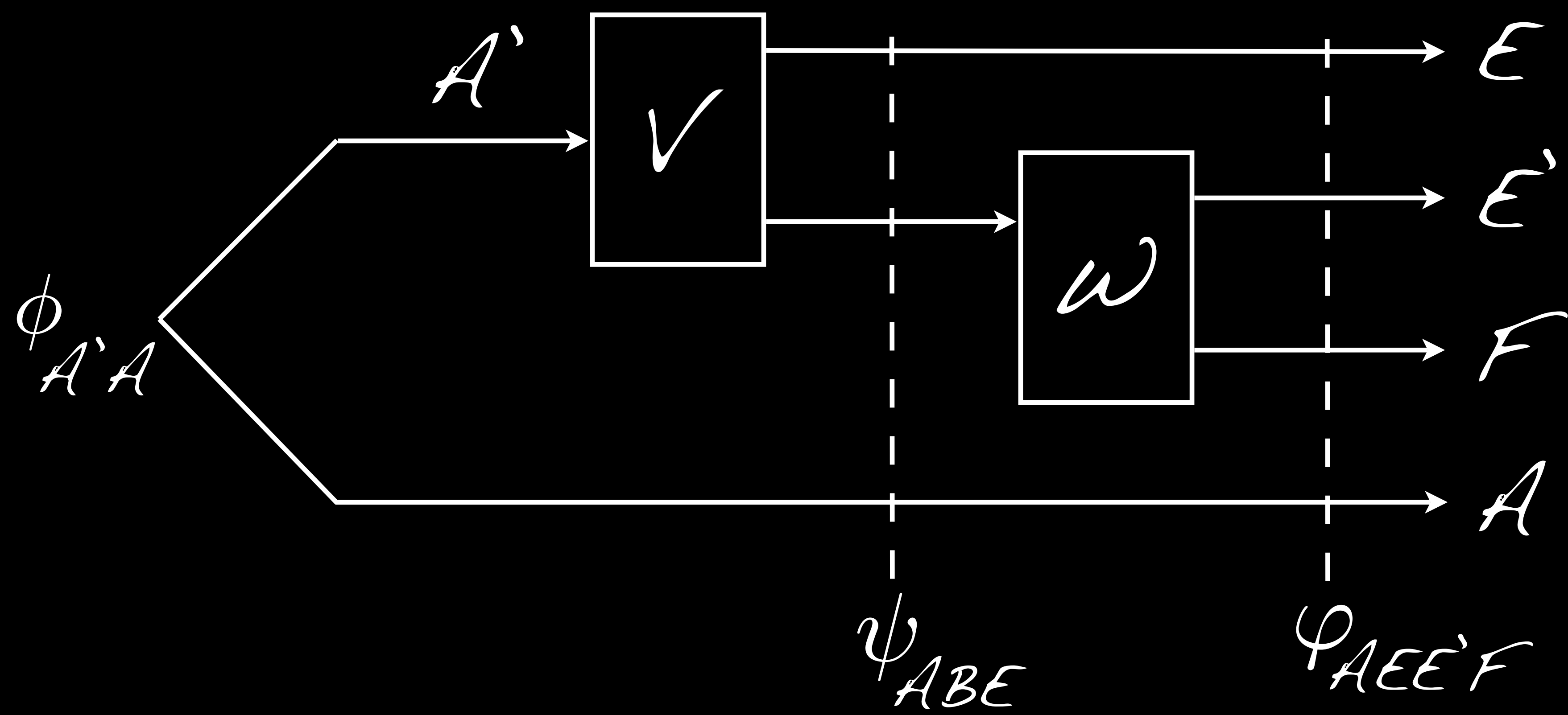


Why we are interested in degr. channels:

$Q^{(1)}(N)$ is additive and so $Q(N) = Q^{(1)}(N)$,

and the latter is a convex optimisation.

[Devetak/Shor, CMP 256:287 (2005)]



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(For anti-degradable N: $Q(N) = 0$.)

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Note: Already for symmetric (degradable & anti-degradable) channels - for which also $Q(N)=0$ - not clear at all.

Exercise: Strong converse for noiseless qubit id_2, even assisted by classical communication.

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Implies: Error goes to one for rates above $E_C(N)$, the entanglement cost of simulating the channel with free classical communication [Berta et al., IEEE-IT 59(10):6779-6795 (2013)].

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Still doesn't take care of 50% erasure channel, dephasing channels, etc!

Thm (Morgan/AW, 1301.4927): For any degradable channel N, all codes with rate $R > Q(N)$ have error at least 0.707, asymptotically.

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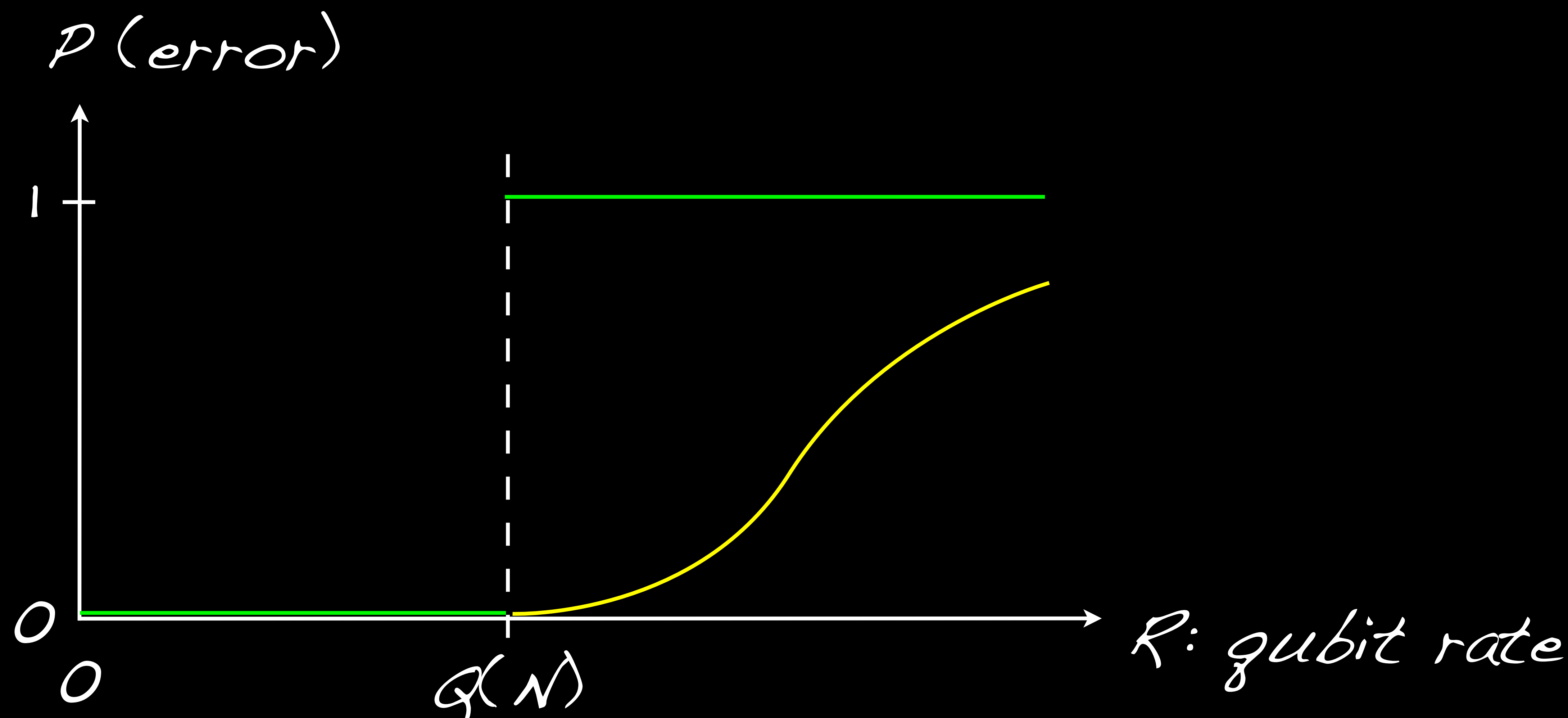

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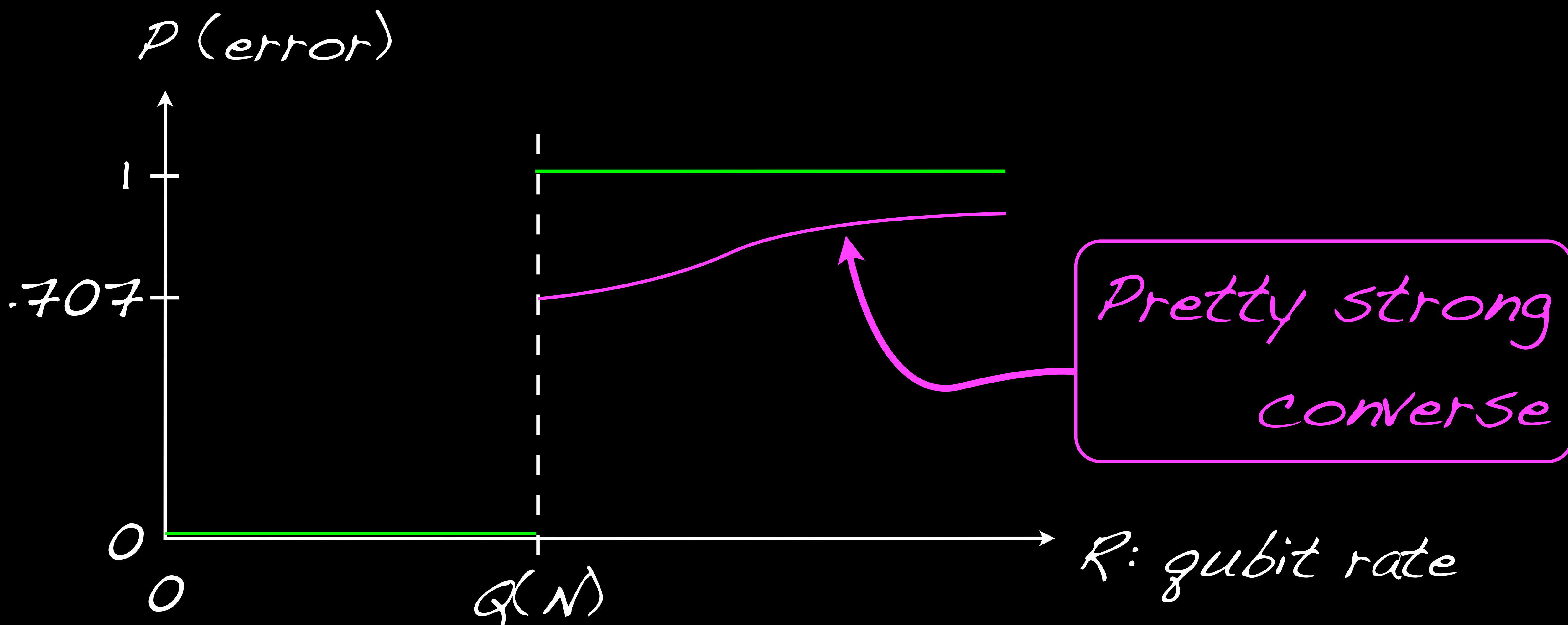

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On the other hand: For larger error, any i.i.d. symmetric channel allows coding of $k = c\sqrt{n}$ qubits, by random codes. More?

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Similar result for private capacity:

Thm (1301.4927): For degradable channel $\mathcal{N}$, if decoding error and distance from perfect privacy are both below some universal threshold, then the rate is asymptotically bounded by $P(\mathcal{N}) = Q(\mathcal{N})$.

Thm: For any degradable channel N, codes with rate $R > Q(N)$ have error at least 0.707, asymptotically.

Significance of symmetric channels:

Thm (1301.4927): If symmetric channels (whose quantum capacity is 0) obey a strong converse, then so do all degradable channels N: for error below 1, rate would be asymptotically bounded by $Q(N)$.

Proof uses tight finite block length characterization of P and Q via (smooth) min-entropies & some tricks: symmetrization, de Finetti theorem, asymptotic equipartition property...

[Cf. R. Renner, PhD thesis, quant-ph/0512258 & M. Tomamichel, PhD thesis, arXiv:1203.2142]

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Can be viewed as a complicated version of the proof of additivity: $P(N)=Q(N)=Q^{(1)}(N)$ for degradable N... :-/

[Devetak/Shor, CMP 256:287 (2005)]

7. Conclusion (sort of...)

- Lesson: To get more precise under-standing of code performance have to abandon von Neumann entropy and embrace non-standard entropies (Rényi entropy, min-entropy, ...)
- Price to pay: Each channel and each capacity requires its own approach. Many open - e.g. multi-user channels...

7. Conclusion (sort of...)

- The trick with the sandwiched channel reduces the additivity of $\chi(N)$ to that of the minimum output Rényi entropy of an associated family of cp (trace non-preserving) maps. Can it be applied to other channels? Other divergences?
  [Wilde/AW/Yang, 1306.1586]
- Can we also get "2nd order" behaviour?
  [Cf. Tomamichel/Tan, 1308.6503 for cq-channels]

7. Conclusion (sort of...)

- Big open problem: "pretty strong" is pretty ugly - how to get full strong converse for Q of degradable channels!? Bottleneck are the symmetric channels, e.g. 50% erasure channel...
- How to prove strong converses without additivity? Note that neither P, Q nor $P^{(1)}$, $Q^{(1)}$, χ are generally additive! (Not known for C.)

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$$\tilde{D}(\rho \| \sigma) \geq \tilde{D}(\mathcal{N}(\rho) \| \mathcal{N}(\sigma)) \geq 0. \qquad (*)$$

Relative entropy

$$D(\rho\|\sigma) = \mathrm{Tr}\,\rho(\log\rho - \log\sigma)$$

is a special case of a whole family of "generalised divergences".

[Cf. Petz, 0909.3647; Müller-Lennert et al., 1306.3142]

Fundamental property is monotonicity: for any cptp map N,

$$\tilde{D}(\rho\|\sigma) \geq \tilde{D}(N(\rho)\|N(\sigma)) \geq 0. \qquad (*)$$

Notation: for binary distributions $P=(p,1-p)$ and $Q=(q,1-q)$, write $\tilde{D}(P\|Q) = \tilde{D}(p\|q)$.

Assume furthermore that

$$\tilde{D}\Big(\bigoplus_x p_x \rho_x \,\Big\|\, \bigoplus_x p_x \sigma_x\Big) = \sum_x p_x \tilde{D}(\rho_x \| \sigma_x). \quad (+)$$

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\tilde{D}(1-\varepsilon \,\|\, 1/M) &\overset{(*)}{\leq} \tilde{D}(\rho_{XB} \| \rho_X \otimes \sigma) \\
&\overset{(+)}{\leq} \frac{1}{M} \sum_m \tilde{D}(\mathcal{N}(\rho_m) \| \sigma) \\
&\leq \max_\rho \tilde{D}(\mathcal{N}(\rho) \| \sigma) =: \chi_{\tilde{D},\sigma}(\mathcal{N})
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[Cf. Nagaoka (≈2000); Polyanskiy/Verdú (2010); Sharma/Warsi, 1205.1712.]

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Sandwiched α-Rényi relative entropy $(\alpha > 1)$

$$\tilde{D}_{\alpha}(\rho \,\|\, \sigma) := \frac{1}{\alpha - 1} \log \operatorname{Tr} \left(\sigma^{\frac{1-\alpha}{2\alpha}} \rho \, \sigma^{\frac{1-\alpha}{2\alpha}} \right)^{\alpha}$$

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[Cf. Müller-Lennert et al., 1306.3142; Beigi 1306.5920; Frank/Lieb 1306.5358]

It's monotonic, has property (+) and is $\leq D_{\alpha}(\rho \,\|\, \sigma) := \frac{1}{\alpha - 1} \log \mathrm{Tr}\, \rho^{\alpha} \sigma^{1-\alpha}$, with which it coincides when states commute.

$$\boxed{\tilde{D}_\alpha(\rho \| \sigma) := \frac{1}{\alpha - 1} \log \operatorname{Tr}\left(\sigma^{\frac{1-\alpha}{2\alpha}} \rho\, \sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha}$$

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Lhs: $\tilde{D}_\alpha(1-\varepsilon \| 1/M) \geq \log M + \frac{\alpha}{\alpha - 1} \log(1-\varepsilon)$

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Lhs: $\tilde{D}_\alpha(1-\varepsilon\|1/M) \ge \log M + \frac{\alpha}{\alpha-1}\log(1-\varepsilon)$

Crucial: $-\chi_{\alpha,\sigma}(\mathcal{N})$ is the minimum α-Rényi output entropy of a perturbed cp map $\mathcal{N}'$,

$$\mathcal{N}'(\rho) = \sigma^{\frac{1-\alpha}{2\alpha}}\mathcal{N}(\rho)\sigma^{\frac{1-\alpha}{2\alpha}}.$$

Have:

$$\log(1-\varepsilon) \leq \left(1-\frac{1}{\alpha}\right)\left(\chi_{\alpha,\sigma}(N) - \log M\right)$$

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Key observation: Sandwiched channel is $(\mathcal{N}')^{\otimes n}$, and $\mathcal{N}'$ is EB if $\mathcal{N}$ is.

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Now apply this to $N^{\otimes n}$, $\sigma^{\otimes n}$, and $M = 2^{nR}$.

Key observation: Sandwiched channel is $(N')^{\otimes n}$, and N' is EB if N is.

$\Rightarrow$ Additivity, $\chi_{\alpha,\sigma}(N^{\otimes n}) = n\,\chi_{\alpha,\sigma}(N)$.

(Because of identity with min output entropy of N')

[King, QIC 2003; Holevo, Russ. Math. Surveys 2006]

Get, for n uses of $\mathcal{N}$ at rate R:

$$\log(1-\varepsilon) \leq n\left(1-\frac{1}{\alpha}\right)\left(\chi_{\alpha,\sigma}(\mathcal{N}) - R\right). \qquad (\&)$$

Get, for n uses of N at rate R:

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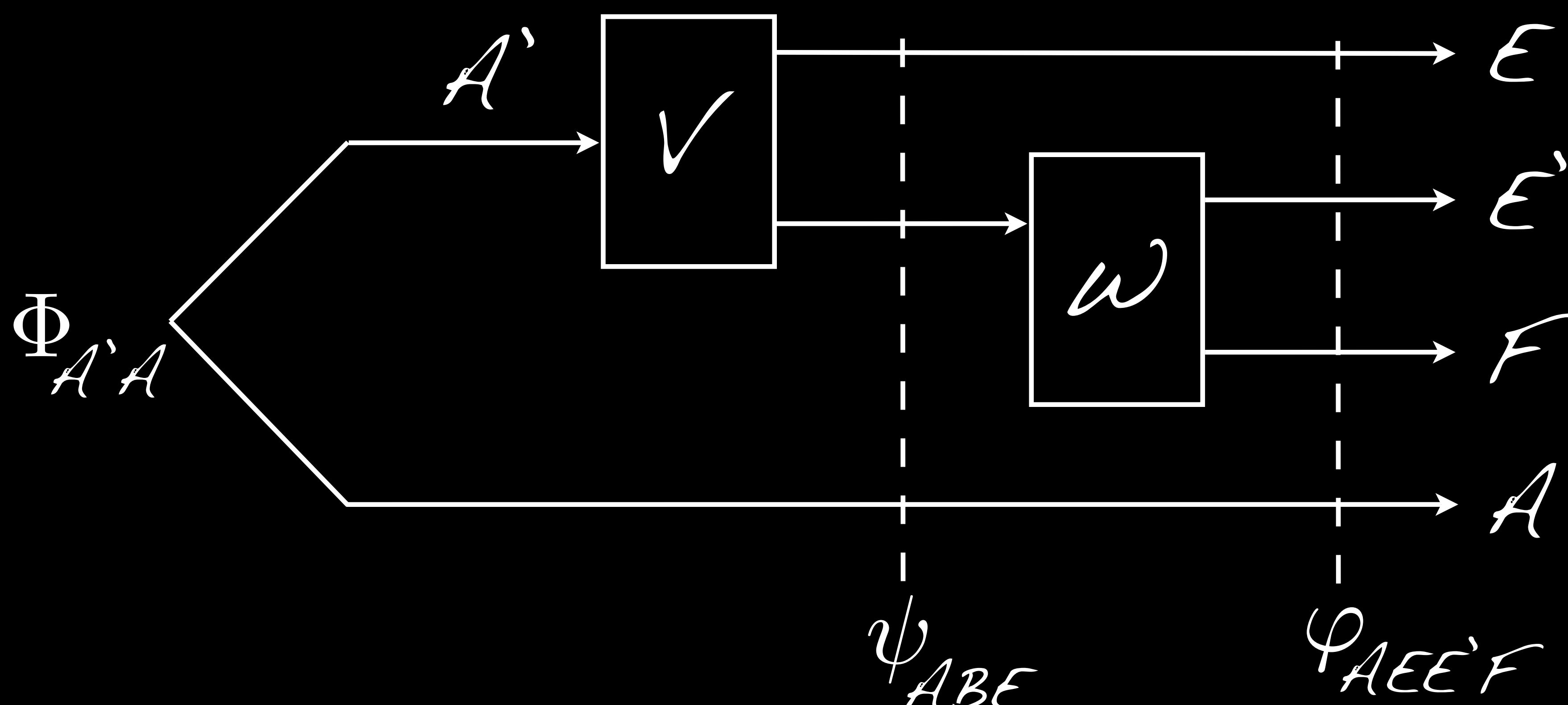
Takes care of EB channels; $\forall$ similar but requires another small trick (...) QED

B. Proof ideas for Q & P

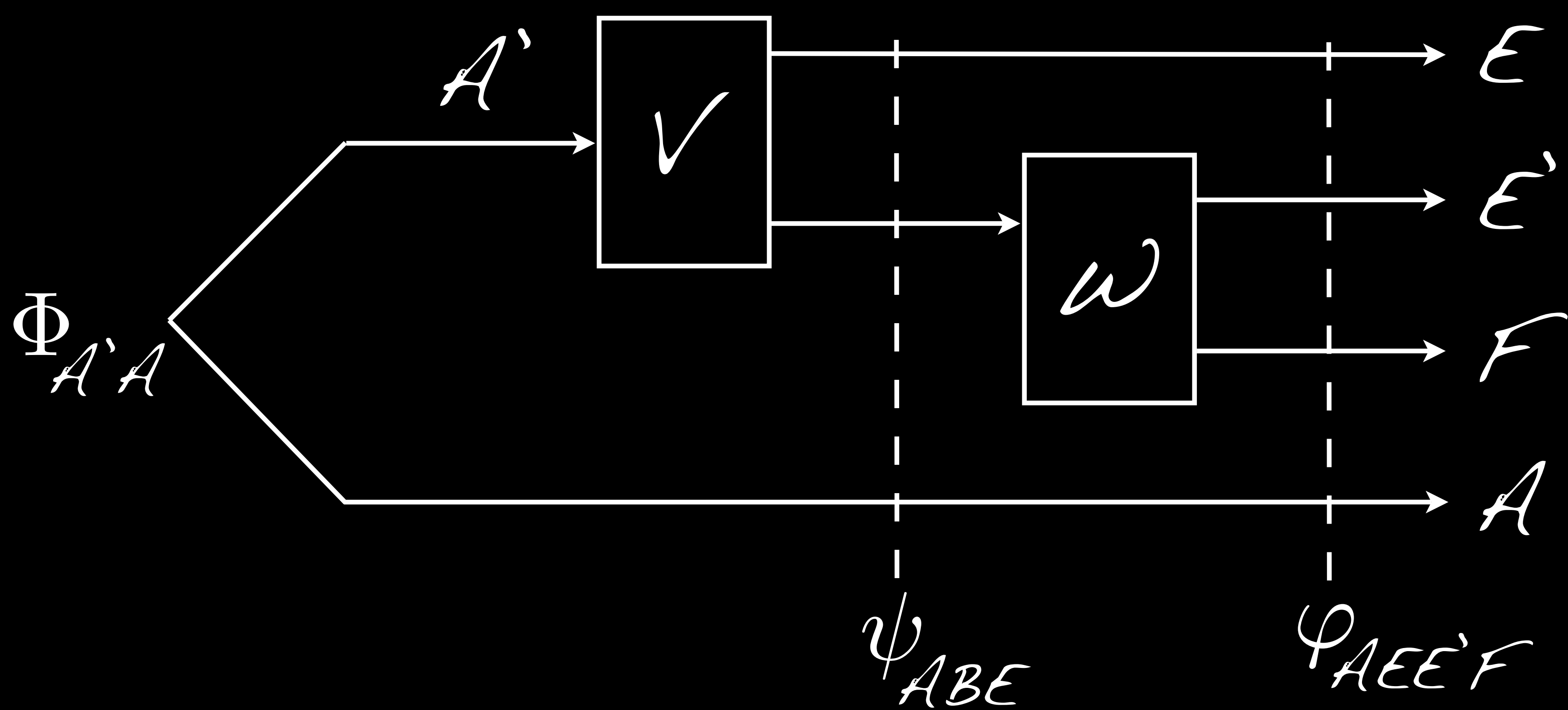
(smooth) min-entropies,
symmetrisation, de Finetti theorem,
AEP

Ideas: (smooth) min-entropies,
symmetrisation, de Finetti theorem,
AEP

1) Use code - for simplicity subspace - with maximally entangled state Φ of k qubits:



Maximally entangled state Φ of k qubits:



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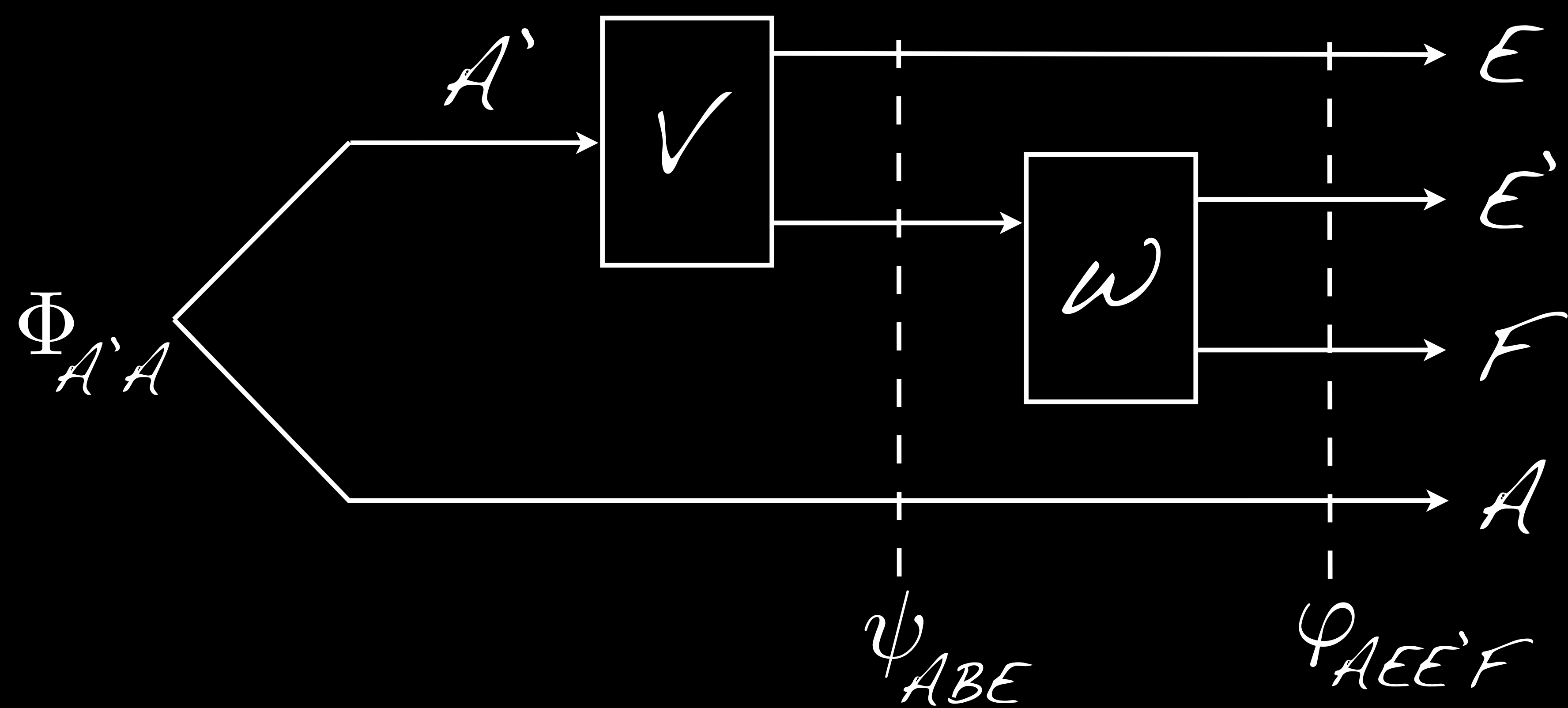

$$k \leq H_{min}^{\epsilon}(A|E)$$

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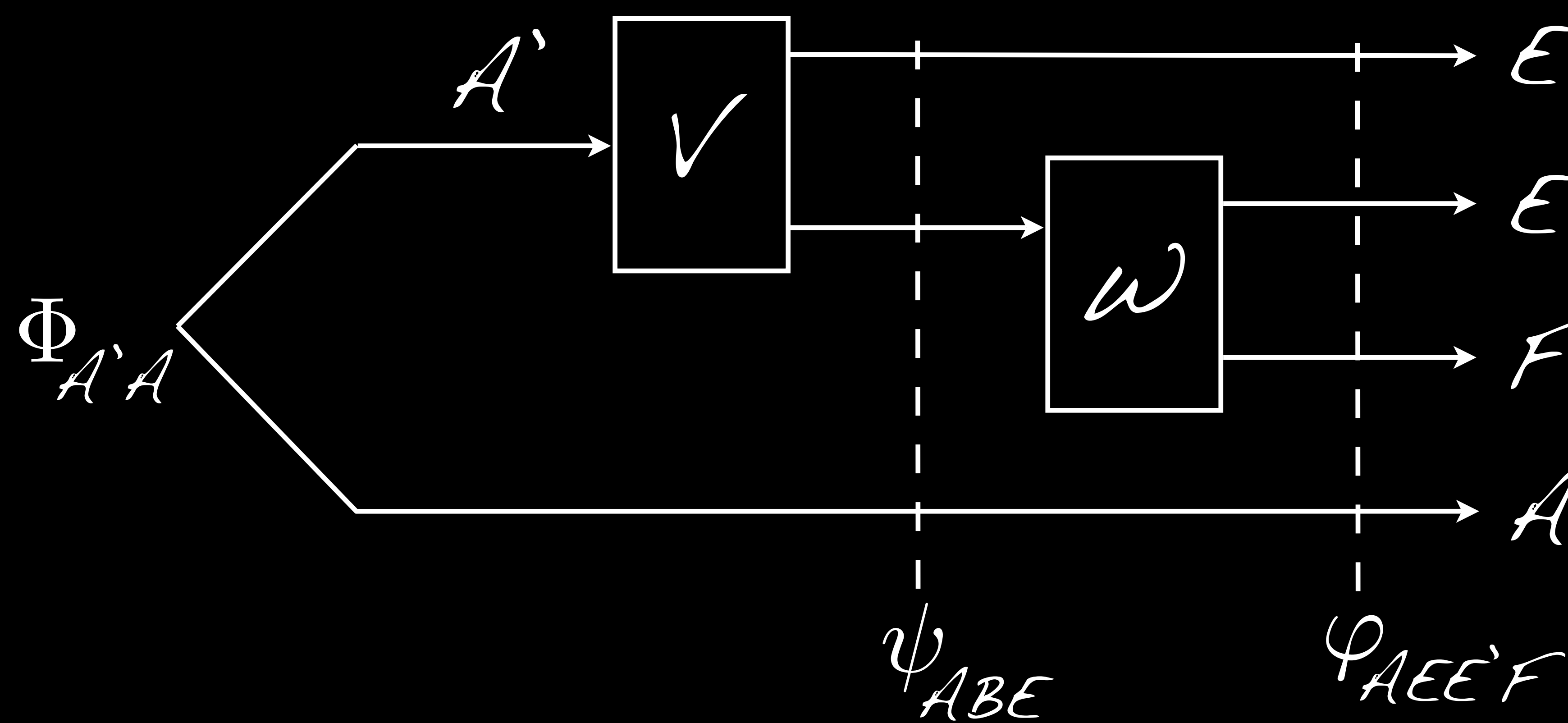

$$k \leq H_{min}^{\epsilon}(A|E) = -H_{max}^{\epsilon}(A|E'F)$$

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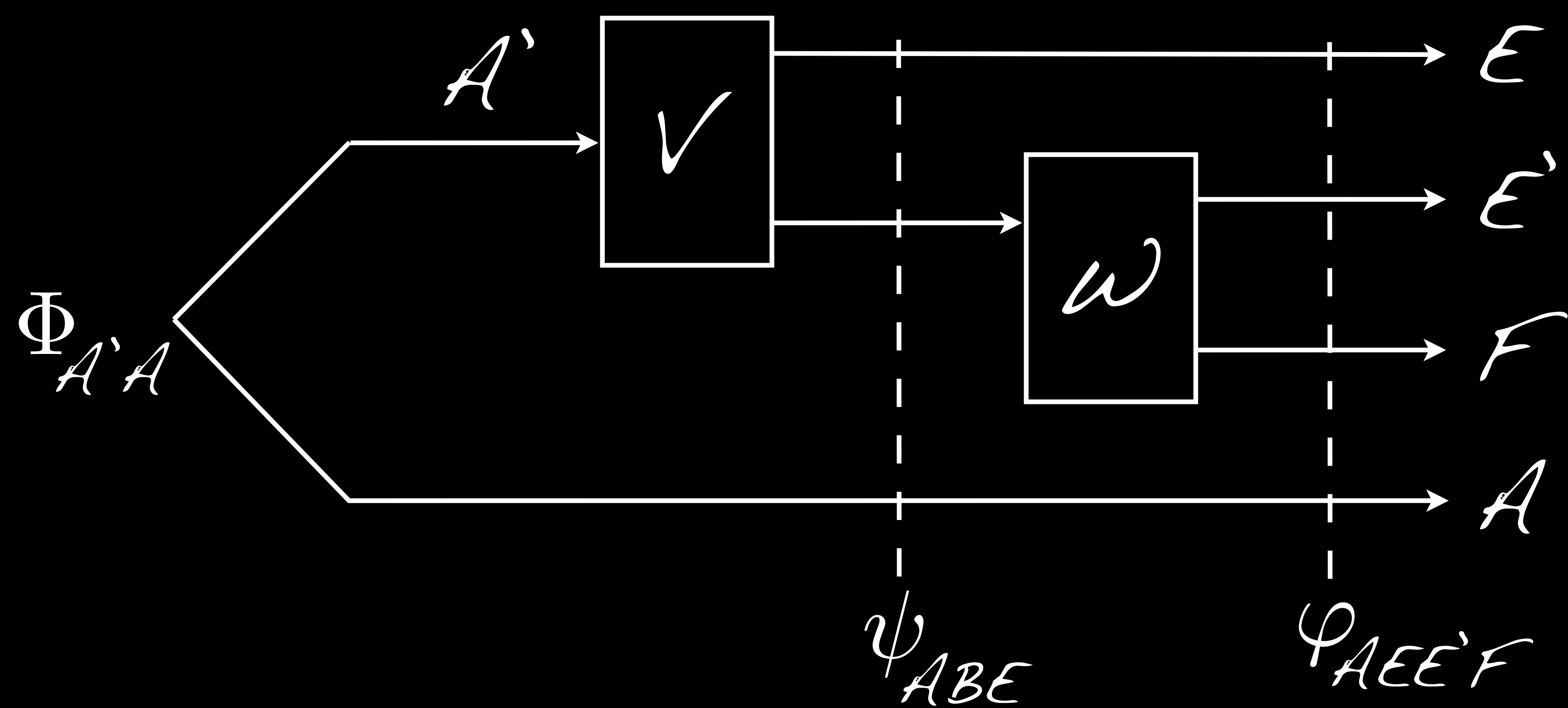

$$k \leq H^{\epsilon}_{min}(A|E) = -H^{\epsilon}_{max}(A|E'F)$$

[For min-entropy calculus, consult
R. Renner, PhD thesis, quant-ph/0512258
& M. Tomamichel, PhD thesis, arXiv:1203.2142]

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[Cf. also Buscemi/Datta, IEEE-IT 56(3), 2010; Datta/Hsieh, 1103.1135]

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Note: If we knew that for n channel uses, the maximum min-entropy is attained on a tensor product input, we'd be done by AEP (= asymptotic equipartition property)...

$$
\begin{aligned}
k &\le H_{min}^{\epsilon}(A|E) \\
&= -H_{max}^{\epsilon}(A|E'F) \\
&\le H_{max}^{\lambda}(F|E') - H_{max}^{\delta}(AF|E') + O(1)
\end{aligned}
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Chain rule, $\delta = \epsilon + 3\lambda$.

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...if $\delta < 0.707$, by inequality H_{min} vs. H_{max}, and using symmetry between E and E'...

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W.r.t. a permutation symmetric input state and $\lambda' = \lambda / \sqrt{2}$

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3) By de Finetti theorem [R. Renner, PhD thesis, quant-ph/0512258]:

$$k \le \max_{\rho_A} H_{max}^{\lambda''}(F^n|E'^n)_{\rho^{\otimes n}} + o(n)$$

4) By AEP (asymptotic equipartition property) [M. Tomamichel, arXiv:1203.2142]:

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QED