Relative entropy and squashed entanglement

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The properties and relations of various existing entanglement measures are very important for our understanding of quantum entanglement. Despite considerable achievements, a lot of issues still remain unclear, even in the bipartite case [1]. Here, we are interested in two families of entanglement measures: *Squashed entanglement* [2–4]

$$E_{sq}(\rho_{AB}) := \inf\left\{\frac{1}{2}I(A;B|E)_\rho : \rho_{ABE} \text{ is an extension of } \rho_{AB}\right\},$$

and the *relative entropy of entanglement* $E_r(\rho_{AB}) := \min\limits_{\sigma_{AB}\in\mathrm{SEP}} D(\rho\|\sigma)$ [5, 6], as well as their analogues and variants.

For each positive operator-valued measurement (POVM) $\{M_i\}_i$, it can be alternatively identified with a measurement operation $\mathcal{M}$, which is a completely positive map from density matrices to probability vectors, $\mathcal{M}(\omega) = \sum_i |i\rangle\langle i| \operatorname{Tr}(\omega M_i)$. On composite system AB, we define some restricted classes of measurements LO, 1-LOCC, LOCC, SEP and PPT. LO, 1-LOCC and LOCC are the sets of measurements that can be implemented by means of local operations, local operations and one-way classical communication, local operations and arbitrary two-way classical communication, respectively; SEP and PPT are the classes of measurements whose POVM elements are separable or positive-partial-transpose, respectively.

We also introduce the variants of relative entropy of entanglement, which will be involved intensively later. Piani defined the relative entropy of entanglement with respect to the set of states G and the restricted class of measurements M [7], as

$$E_{r,\mathsf{M}}^{(\mathrm{G})}(\rho) := \inf_{\sigma\in\mathrm{G}} \sup_{\mathcal{M}\in\mathsf{M}} D\big(\mathcal{M}(\rho)\|\mathcal{M}(\sigma)\big). \tag{1}$$

In this paper, G is usually the set of separable states SEP. Therefore, we abbreviate $E_{r,\mathsf{M}}^{(\mathrm{SEP})}$ to $E_{r,\mathsf{M}}$ for simplicity. Besides, regularization of an entanglement measure f is defined as $f^\infty(\rho_{AB}) := \lim\limits_{n\to\infty} \frac{1}{n} f(\rho_{AB}^{\otimes n})$.

I. OUR RESULTS

Monogamy relation for relative entropy of entanglement. One of the most fundamental properties of entanglement is monogamy: the more a quantum system is entangled with another, then the less it is entangled with the others. For any entanglement measure f, one would expect a quantitative characterization of monogamy of the form $f(\rho_{1:23}) \geq f(\rho_{1:2}) + f(\rho_{1:3})$. Although this is really the case for squashed entanglement [8], relative entropy of entanglement – along with many other entanglement measures – does not satisfy such a strong relation, with the antisymmetric state being a counterexample [9, 10].

Here, we propose and prove a properly weakened monogamy inequality for relative entropy of entanglement, by invoking its one-way LOCC variant.

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Theorem 1 *For every tripartite quantum state ρ_{ABE}, we have*

$$E_r(\rho_{B:AE}) \geq E_{r,\text{1-LOCC}}(\rho_{AB}) + E_r^{\infty}(\rho_{BE}), \tag{2}$$
$$E_r^{\infty}(\rho_{B:AE}) \geq E_{r,\text{1-LOCC}}^{\infty}(\rho_{AB}) + E_r^{\infty}(\rho_{BE}). \tag{3}$$

Eq. (3) is obtained from Eq. (2) by regularizing both sizes, and it becomes stronger due to the subadditivity of E_r and superadditivity of $E_{r,\text{1-LOCC}}$ [7, 11].

It is worth mentioning that Eq. (2) and Eq. (3) are in the form similar to Piani's superadditivity-like relation

$$E_r(\rho_{A_1A_2:B_1B_2}) \geq E_{r,\mathsf{M}}(\rho_{A_1B_1}) + E_r(\rho_{A_2B_2}),$$

with M be LOCC or SEP. The difference is that in our result, there is only one single system B on the left side, while it appears twice on the right side. As a result, the price we have to pay is degrading the measurement class to 1-LOCC and imposing a regularization in the two terms of the right side, respectively (see Eq. (2)). One the other hand, our proof needs new technique, which is derived in the context of quantum hypothesis testing under restricted measurement class 1-LOCC.

Commensurate lower bound for squashed entanglement. Following the monogamy relation, we provide a commensurate and faithful lower bound for squashed entanglement. Instead of the one-way LOCC trace distance bound in [12], our result is in the form of one-way LOCC relative entropy of entanglement, which is more natural and stronger.

Theorem 2 *For any quantum state ρ_{AB}, we have*

$$E_{sq}(\rho_{AB}) \geq \frac{1}{2}E_{r,\text{1-LOCC}}^{\infty}(\rho_{AB}) \geq \frac{1}{2}E_{r,\text{1-LOCC}}(\rho_{AB}). \tag{4}$$

The core inequality for von Neumann entropy, strong subadditivity [13], states that for any tripartite state ρ_{ABE},

$$I(A;B|E)_\rho \geq 0.$$

Recalling the definition of squashed entanglement, Theorem 2 implies

$$I(A;B|E)_\rho \geq E_{r,\text{1-LOCC}}(\rho_{AB}),$$

and hence strengthens the strong subadditivity inequality by relating it to a distance-like entanglement measure on two of the subsystems.

On the one hand, applying Pinsker's inequality [14], we are able to recover the trace-distance bound of [12], even with a slightly better constant factor:

$$E_{sq}(\rho_{AB}) \geq \frac{1}{4\ln 2} \min_{\sigma_{AB}\in\text{SEP}} \|\rho_{AB} - \sigma_{AB}\|_{\text{1-LOCC}}^2 .$$

On the other hand, while the trace-distance bound can be at most $O(1)$, our new bound (4) can be very large. Indeed, $E_{r,\text{1-LOCC}}$ is asymptotically normalized, in the sense of Proposition 4.

Asymptotic continuity. To quantify the resources in quantum protocols in a physically robust way, entanglement measures are expected to be asymptotically continuous. Piani's paper [7] contains the proofs of several properties of $E_{r,\mathsf{M}}^{(\mathrm{G})}$ for certain combination of G and M. Now we also show asymptotic continuity under very general conditions.

We say that a set S is star-shaped with respect to some $x_0 \in \mathrm{S}$, if $px + (1-p)x_0 \in \mathrm{S}$ for all $x \in \mathrm{S}$ and $0 \leq p \leq 1$.

Proposition 3 *Let* M *be any set of measurements, and* G *be a set of states on a quantum system with Hilbert space dimension k, containing the maximally mixed state τ and such that in fact* G *is star-shaped with respect to τ. Let ρ, ρ' be two states of the quantum system with $\|\rho - \rho'\|_{\mathsf{M}} \leq \epsilon \leq \frac{1}{e}$. Then,*

$$\left|E_{r,\mathsf{M}}^{(\mathrm{G})}(\rho) - E_{r,\mathsf{M}}^{(\mathrm{G})}(\rho')\right| \leq 2\epsilon \log \frac{6k}{\epsilon}.$$

Evaluation on maximally entangled states and pure states. The entanglement measure $E_{r,\mathsf{M}}$ is difficult to calculate due to the two optimizations in its definition. Here we conduct the first exact evaluation on maximally entangled states, with M be any of $\{\mathsf{LO}, \mathsf{1\text{-}LOCC}, \mathsf{LOCC}, \mathsf{SEP}, \mathsf{PPT}\}$. The basic idea is to make use of the symmetry of $\frac{1}{\sqrt{d}}\sum_{i=1}^{d}|ii\rangle$, namely, invariance under unitary operation $U \otimes \overline{U}$. Then, with the help of asymptotic continuity of Proposition 3, we further obtain their regularized versions on general pure states.

At first glance, the restricted class of measurements M may make $E_{r,\mathsf{M}}$ much smaller than the normal relative entropy of entanglement. However, in our case we find that they are almost the same when the local dimension is very large.

Proposition 4 *For the rank-d maximally entangled state* Φ_d*,*

$$E_{r,\mathsf{LO}}(\Phi_d) = E_{r,\mathsf{1\text{-}LOCC}}(\Phi_d) = E_{r,\mathsf{LOCC}}(\Phi_d) = E_{r,\mathsf{SEP}}(\Phi_d) = E_{r,\mathsf{PPT}}(\Phi_d) = \log(d+1) - 1. \tag{5}$$

As a corollary, this implies that for pure state ψ_{AB}*, the regularized versions are equal to the entropic pure state entanglement:*

$$E^{\infty}_{r,\mathsf{LO}}(\psi_{AB}) = E^{\infty}_{r,\mathsf{1\text{-}LOCC}}(\psi_{AB}) = E^{\infty}_{r,\mathsf{LOCC}}(\psi_{AB}) = E^{\infty}_{r,\mathsf{SEP}}(\psi_{AB}) = E^{\infty}_{r,\mathsf{PPT}}(\psi_{AB}) = S(\operatorname{Tr}_B \psi). \tag{6}$$

Comparisons between entanglement measures. We consider comparisons and separations between some important entanglement measures and obtain several new results. These new relations, together with previous known ones, are summarized in Fig. 1.

Here, $E_{r,\leftrightarrow}(\rho_{AB}) := E_{r,\mathsf{LOCC}}(\rho_{AB})$ and $E_{r,\rightarrow}(\rho_{AB}) := \sup\{E_{r,\mathsf{1\text{-}LOCC}}(\Lambda(\rho_{AB})) : \Lambda \text{ being LOCC}\}$. Note that $E_{r,\rightarrow}$ can be regarded as an "update" of $E_{r,\mathsf{1\text{-}LOCC}}$ such that it is LOCC monotone. E_I is the conditional entanglement of mutual information [15], and $E_{sq,c}$ is the c-squashed entanglement [16]. The others are familiar notions: E_f is the entanglement of formation; $E_c = E_f^{\infty}$ is the entanglement cost; K_d is the distillable key and E_d the distillable entanglement.

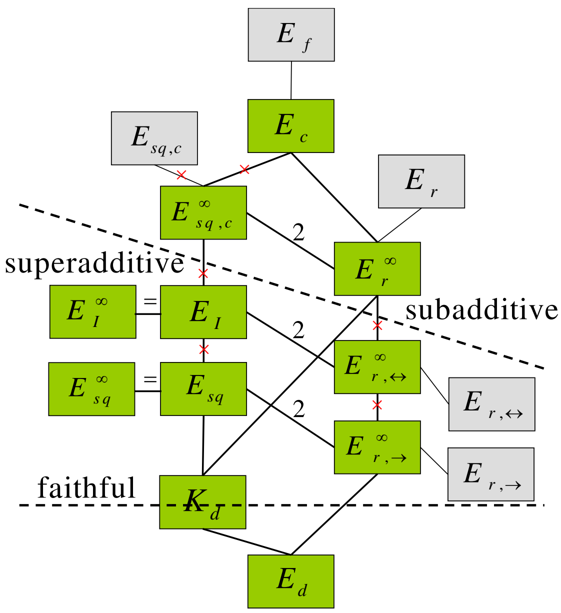

FIG. 1: Relations between some entanglement measures. When two quantities are connected by a line with a constant above (constant 1 is omitted), it means that the higher one multiplied by the constant is no smaller than the lower one. For those entanglement measures of which the separation is still unknown, we mark a red cross on the line that connects them. The upper dashed line divides these entanglement measures into two groups: the upper ones are subadditive and the lower ones are superadditive. Entanglement measures above the lower dashed line are faithful, while the only one below this line, E_d, is not faithful [17]. Whether the distillable key, K_d, is faithful or not, is still an open question. Hence, we put the line on it.

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