

Negative Quasi-Probability as a Resource for Quantum Computation

While it is widely believed that quantum computers can solve certain problems with exponentially fewer resources than their classical counterparts, the scope of the physical resources of the underlying quantum systems that enable universal quantum computation is not well understood. For example, for the standard circuit model of quantum computation, Vidal has shown that high-entanglement is necessary for an exponential speed-up [21]; however, it is also known that access to high-entanglement is not sufficient [11]. Moreover, in alternative models of quantum computation such as DQC1 [15], algorithms that may be performed on highly-mixed input states appear to be more powerful than classical computation even though there appears to be a negligible amount of entanglement in the underlying quantum system [4]. This suggests that large amounts of entanglement, purity or even coherence may not be necessary resources for quantum-computational speed-up. One of the central open problems of quantum information is to understand which sets of quantum resources are jointly necessary and sufficient to enable an exponential speed-up over classical computation. Any solution to this important problem may point to more practical experimental means of achieving the benefits of quantum computation.

The question of whether a restricted subset of quantum theory is still sufficient for a given task is meaningful when there is a specific context that divides the full set of possible quantum operations into two classes: the restricted subset of operations that are accessible or easy to implement and the remainder that are not. In such a context it is then natural to consider the difficult operations as resources and ask how much, if any, of these resources are required. For example, a common paradigm in quantum communication is that of two or more spatially separated parties for which local quantum operations and classical communication define a restricted set of operations that are accessible or “free resources”, whereas joint quantum operations are not free; in this context entanglement is the natural resource for quantum *communication*. Astonishingly, there is not yet a corresponding resource theory for the task of quantum *computation*.

The major obstacle to physical realizations of quantum computation is that real world devices suffer random noise when they execute quantum algorithms. Fault tolerant quantum computation offers a partial resolution to this problem by allowing the effective error rates on logically encoded computations to be reduced below the error rate of the physical (unencoded) computation. Transversal unitary gates, i.e., gates that do not spread errors within each code block, play a critical role in fault-tolerant quantum computation. Recent theoretical work has shown that a set of quantum gates which is both universal and transversal, and hence fault-tolerant [3, 6, 22], does not exist. That is, any scheme for fault tolerant quantum computation divides quantum operations into two classes: those with a fault-tolerant implementation – these are the “free resources” – and the remainder – these are not free but are required to achieve universality. For a fixed fault tolerant scheme the critical question is: what are necessary and sufficient physical resources to promote fault-tolerant computation to universal quantum computation?

The known fault tolerant schemes with the best performance are built around the well-known stabilizer formalism [12], in which a distinguished set of preparations, measurements, and unitary transformations (the “stabilizer operations”) have a fault tolerant implementation. Stabilizer operations also arise naturally in some physical systems with topological order [5, 16, 17]. As described above, the transversal set of stabilizer operations do not give a universal gate set and must be supplemented with some additional (non-stabilizer) resource. A celebrated scheme for overcoming this limitation is the magic state model of quantum computation devised by Bravyi and Kitaev [2] where the additional resource is a set of ancilla systems prepared in a some (generally noisy) non-stabilizer quantum state. Hence, in this important paradigm, the question of which physical resources are required for universal fault-tolerant quantum computation reduces to the following: which non-stabilizer states are necessary and sufficient to promote stabilizer computation to universal quantum computation?

In this talk I will identify a non-trivial closed, convex subset of the space of quantum states which we have shown is incapable of producing universal fault-tolerant quantum computation[19]. In particular, this convex subset strictly contains the convex hull of stabilizer states, and thereby proves that there exists a class of *bound universal states*, i.e. states that can not be prepared from convex combinations of stabilizer states and yet are not useful for quantum computation. Thus the proof of the existence of bound universal states resolves in the negative the open problem raised by Bravyi and Kitaev [2] of whether *all* non-stabilizer states promote stabilizer computation to universal quantum computation. Furthermore, I will give an efficient simulation algorithm for the subset of quantum theory that consists of operations from the stabilizer formalism acting on inputs from our non-universal region, which includes mixed states both inside and outside the convex hull of stabilizer states. This simulation scheme is an extension of the celebrated Gottesman-Knill theorem [1, 11] to a broader class of input state and should be of independent interest.

Our theoretical method for proving these results is to construct a classical, local hidden variable model for the subtheory of quantum theory that consists of the stabilizer formalism and then determine the scope of additional quantum resources that are also described by this model. Indeed our local hidden variable model is a distinguished quasi-probability representation with non-negative elements. For a d dimensional quantum system there are many

possible ways to represent arbitrary quantum states as quasi-probability distributions over a phase space of d^2 points and projective measurements as conditional quasi-probability distributions over the same space (see [8, 9] for further details). Perhaps unsurprisingly, it has been shown that the full quantum theory can not be represented with non-negative elements in any such representation [7–9, 18]. However, one might expect that a subtheory of quantum theory that is inadequate for quantum speed-up might be represented non-negatively, i.e. as a true probability theory, in some natural choice of quasi-probability representation. For the context described above, we seek a quasi-probability representation reflecting our natural operational restriction, in particular, we require that stabilizer states and projective measurements onto stabilizer states have non-negative representation and that unitary stabilizer operations (i.e., Clifford transformations) correspond to stochastic processes. Conveniently, for quantum systems with odd Hilbert space dimension such a representation is already known to exist: this is the discrete Wigner function picked out by Gross [13, 14] from the broad class defined by Gibbons *et al* [10]. In such a representation it is natural to examine whether the resources that are necessary or sufficient for quantum speed-up correspond to those that are not represented by non-negative elements of the representation.

With this insight in hand the results covered by this talk may now be stated more carefully:

Classically efficient simulation of positive Wigner functions: The set of fault tolerant quantum logic gates in the stabilizer formalism are known as the *Clifford* gates. Our first contribution is an explicit simulation protocol for quantum circuits composed of Clifford gates acting on input states with positive discrete Wigner representation. We also allow arbitrary product measurements with positive discrete Wigner representation. This simulation is efficient (linear) in the number of input registers to the quantum circuit. This simulation scheme is an extension of the celebrated Gottesman-Knill theorem and should be of independent interest.

Negativity is necessary for magic state distillation: This simulation protocol implies that states outside the stabilizer formalism with positive discrete Wigner function (bound universal states) are not useful for magic state distillation. I will give a direct proof of this fact exploiting only the observation that negative discrete Wigner representation can not be created by stabilizer operations. This proof has a more general range of applicability than the efficient simulation scheme and also makes clear the conceptual importance of negative quasi-probability as a *resource* for stabilizer computation.

Geometry of positive Wigner functions: The set of quantum states with positive discrete Wigner function strictly contains the set of (convex combinations of) stabilizer states. To prove this fact I will fully describe the geometry of the region of quantum state space with positive discrete Wigner representation. Concretely, the facets of the classical probability simplex defining the discrete Wigner function are also facets of the polytope with the (pure) stabilizer states as its vertices. Since there are many more facets of the stabilizer polytope than of the simplex this suffices to show the existence of non-stabilizer states with positive representation.

Extension to Quantum Optics: The above results have been derived for finite dimensional systems. There is a strong analogy between the finite dimensional stabilizer formalism and linear optics as well as between the discrete Wigner function and the continuous Wigner function. This analogy can be formalized by showing that the simulation protocol outlined for the stabilizer formalism translates naturally (albeit with significant technical challenges) to a simulation protocol for quantum computation making use of linear optical transformations and measurements supplemented with arbitrary state preparations with positive Wigner representation.

Measure of the Magicness of Quantum States: The preceding results are purely binary: if an ancilla preparation is positively represented then it can not promote stabilizer computation to universal computation but if it is even very slightly negatively represented then none of the preceding work applies. It is natural to wonder whether the *amount* of negativity is meaningful measure of how useful a state is for promoting stabilizer operations to full quantum power. In the context of entanglement theory the amount of entanglement of a quantum state can be quantified by “entanglement monotones”, functions mapping quantum states to values on the real line that are non-increasing under LOCC operations. In the context of stabilizer computation we require a “magicness monotone”, a function $f : \mathcal{S}(\mathcal{H}_d) \rightarrow \mathbb{R}$ which is non-increasing under stabilizer operations; i.e. $f(\Lambda(\rho)) \leq f(\rho)$ if Λ is in the stabilizer formalism. I will show that the sum of the negative entries of the Wigner representation of a quantum state (the “sum negativity”) is a magicness monotone. As is the case in entanglement theory, the development of a monotone gives us the tools to address a wide variety of important questions, such as: what resource states are useful for computational speedup? What is the optimal rate at which one resource state can be transformed into another? How well do existing protocols make use of their input resources? What are general principals for the design of protocols that make efficient use of resources?

It is important to bear in mind that these results thus far only apply to systems of qudits with odd Hilbert space dimension as the discrete Wigner function is only defined for such systems. It remains to be established whether this distinction between odd and even Hilbert space dimension is merely mathematical frippery (as in the case of error correction, which requires a similar distinction between bits and dits) or if it reveals something deep about the quantum formalism.

The first three results appear in [19]. The fourth result is established by [20]. The final result is part of work with

Dan Gottesman, Ali Hamed and Joseph Emerson that is still in preparation.

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