Recoupling Coefficients and Quantum Entropies (Extended Abstract of [1])

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In this work, we show that the asymptotic limit of the recoupling coefficients of the symmetric group is characterized by the existence of quantum states of three particles with given eigenvalues for their reduced density matrices. This parallels Wigner's observation that the semiclassical behavior of the $6j$-symbols for SU(2)—fundamental to the quantum theory of angular momentum—is governed by the existence of Euclidean tetrahedra. Interpreted differently, our result characterizes the existence of quantum states with certain marginal eigenvalues in terms of representation theory. We furthermore explain how to deduce solely from symmetry properties of the recoupling coefficients the strong subadditivity of the von Neumann entropy, first proved by Lieb and Ruskai, and discuss possible generalizations of our result.

Spin, the quantum-mechanical analog of angular momentum, is mathematically given by an irreducible representation of the group SU(2), the covering group of the rotations in three-dimensional space. In order to compute the total spin of two particles, one needs to decompose the tensor product of the individual spins into irreducible representations. This decomposition is described by a unitary matrix whose entries are known as the Clebsch–Gordan coefficients, or, equivalently, as the Wigner $3j$-symbols (the latter differ only in normalisation). Clebsch–Gordan coefficients are fundamental to quantum theory, governing, for instance, optical transitions in atoms and molecules. When computing the total spin of three spins j_1, j_2 and j_3, one can either start by decomposing j_1 and j_2 to obtain j_{12} and then further decompose j_{12} and j_3 into j_{123}, or, alternatively, decompose j_2 and j_3 into j_{23} and then decompose j_1 and j_{23} into j_{123}. The entries of the unitary matrix relating these two decompositions are known as the *recoupling coefficients*, or as the *Wigner* $6j$*-symbols* in their rescaled, more symmetric form [2]; they depend only on the six mentioned spins (the Racah W-coefficients [3] are also closely related). The semiclassical limit where all spins are simultaneously rescaled by $k \to \infty$ was first considered by Wigner [2] who noted that the absolute value squared of the $6j$-symbol—corresponding to the probability of particles two and three having total spin j_{23} given the spins $j_1, j_2, j_3, j_{12}, j_{123}$—oscillates around the inverse volume of the tetrahedron whose edges have length equal to the six spins if such a tetrahedron exists; in particular, it then decays polynomially with k. If no such tetrahedron exists then the $6j$-symbol decays exponentially. A more precise formula together with a heuristic proof was given by Ponzano and Regge [4] and only proved in 1999 by Roberts [5]. Wigner $6j$-symbols and their asymptotics have recently been studied in depth in the context of quantum gravity, more precisely, in connection with spin foams and spin networks (see e.g. [6–11]). They also have applications in quantum information theory [12, 13] and quantum computation [14, 15] (cf. [16]).

In this work we consider the *recoupling coefficients of the symmetric group* S_k, defined in direct analogy to the case of SU(2). The spins are replaced by irreducible representations $[\lambda]$ of S_k, labelled by Young diagrams λ with k boxes. Since, in contrast to SU(2), the Clebsch–Gordan series for S_k, $[\alpha] \otimes [\beta] \cong \bigoplus_\lambda [\lambda] \otimes \mathrm{H}_\lambda^{\alpha\beta}$, is not multiplicity-free (i.e., $\dim \mathrm{H}_\lambda^{\alpha\beta} > 1$ in general), the recoupling coefficients are no longer scalars. Instead, they are the linear maps

$$\begin{bmatrix} \alpha & \beta & \mu \\ \gamma & \lambda & \nu \end{bmatrix} \colon \mathrm{H}_\lambda^{\mu\gamma} \otimes \mathrm{H}_\mu^{\alpha\beta} \to \mathrm{H}_{\alpha\nu}^{\lambda} \otimes \mathrm{H}_{\beta\gamma}^{\nu} \tag{1}$$

which make up the isomorphism relating the two decompositions of a triple tensor product

$$[\alpha] \otimes [\beta] \otimes [\gamma] \cong \bigoplus_{\lambda} [\lambda] \otimes \left(\bigoplus_{\mu} \mathrm{H}_{\lambda}^{\mu\gamma} \otimes \mathrm{H}_{\mu}^{\alpha\beta} \right) \cong \bigoplus_{\lambda} [\lambda] \otimes \left(\bigoplus_{\nu} \mathrm{H}_{\lambda}^{\alpha\nu} \otimes \mathrm{H}_{\nu}^{\beta\gamma} \right). \tag{2}$$

We now consider the symbols' Hilbert–Schmidt norm in the limit where k becomes large but where the normalized Young diagrams $\overline{\alpha} := \alpha/k$, $\overline{\beta} := \beta/k$, etc. converge. To state our result, we introduce the following terminology: For a density operator ρ_{ABC} on $\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$, we consider the reduced density operators $\rho_{AB} = \mathrm{tr}_C(\rho_{ABC})$, $\rho_A = \mathrm{tr}_{BC}(\rho_{ABC})$ etc., and denote by r_{ABC}, r_{AB}, r_A, etc., the corresponding vectors of eigenvalues (each ordered non-increasingly, e.g. $r_{ABC,1} \geq r_{ABC,2} \geq \ldots$). We call the tuple $(r_A, r_B, r_C, r_{AB}, r_{BC}, r_{ABC})$ the eigenvalues associated to ρ_{ABC}. The missing r_{AC} will be discussed in the conclusions. The following theorem is the main result of our work:

Theorem 1. *If there exists a quantum state ρ_{ABC} with eigenvalues $(r_A, r_B, r_C, r_{AB}, r_{BC}, r_{ABC})$ then there exist Young diagrams $\alpha, \beta, \gamma, \mu, \nu, \lambda$ with $k \to \infty$ boxes such that*

$$\lim_{k\to\infty} (\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\mu}, \overline{\nu}, \overline{\lambda}) = (r_A, r_B, r_C, r_{AB}, r_{BC}, r_{ABC}) \tag{3}$$

and

$$\| \begin{bmatrix} \alpha & \beta & \mu \\ \gamma & \lambda & \nu \end{bmatrix} \|_{\mathrm{HS}} \geq \frac{1}{\mathrm{poly}(k)}. \tag{4}$$

Conversely, if $(r_A, r_B, r_C, r_{AB}, r_{BC}, r_{ABC})$ is not associated to any tripartite quantum state then for every sequence of Young diagrams satisfying (3) *we have*

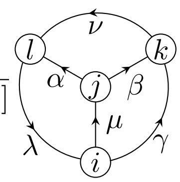
$$\| \begin{bmatrix} \alpha & \beta & \mu \\ \gamma & \lambda & \nu \end{bmatrix} \|_{\mathrm{HS}} \leq \exp(-\Omega(k)). \tag{5}$$

This description of the asymptotics in terms of the existence of a geometric object (here a quantum state with certain spectral properties) can be seen as a direct analog to the existence of Wigner's tetrahedra. Our work can also be understood in the context of the quantum marginal problem: we characterize the existence of quantum states of three particles with certain marginal eigenvalues. This extends significantly recent results where it was shown that the case of two particles is guided by the asymptotics of the Kronecker coefficients of S_k (i.e., the dimensions of the multiplicity spaces $H_{\lambda}^{\alpha\beta}$) [17–21], and ties in with the current interest in joint typicality for multipartite quantum systems [22, 23].

The proof of Theorem 1 builds on and generalizes the quantum information methods developed in [17, 24, 25]. We work with Hilbert space $(\mathbb{C}^{abc})^{\otimes k}$ in which both the symmetric group S_k and tripartite quantum states are at home. Using Schur–Weyl duality, we can express the recoupling coefficients in terms of the overlap of two incompatible (i.e., non-commuting) decompositions into irreducibles, and the connection to quantum states uses the spectrum estimation theorem [24] (cf. [17, 26, 27]), which says that k copies of a quantum state ρ on $\mathbb{C}^d$ are mostly supported on the irreducible representations satisfying $\overline{\lambda} = \lambda/k \approx \mathrm{spec}\,\rho = r$. Finally, in the proof of the converse we also use the post-selection technique [25, 28]. The inherent non-commutativity of the tripartite setup—not present in the bipartite situation—is the main mathematical challenge we needed to overcome (it is as well the central obstacle for multipartite joint typicality). Geometrically, the study of recoupling coefficients involves moment maps for non-commuting group actions, and hence the algebro-geometric methods of [18, 19, 29] are not applicable; our representation-theoretic characterization of spectra therefore goes beyond these results.

We now describe how to prove the strong subadditivity of the von Neumann entropy using Theorem 1 and symmetry considerations. For this, we note that the recoupling coefficients can be conveniently expressed in the graphical notation of the fusion basis states, commonly employed in topological quantum computation (see e.g. [30, 31]):

$$\begin{bmatrix} \alpha & \beta & \mu \\ \gamma & \lambda & \nu \end{bmatrix}^{kl}_{ij} = \frac{1}{\dim[\lambda]} \qquad (6)$$



The right-hand side diagram can also be interpreted as the contraction of a tensor network built from four Clebsch–Gordan transformations. Using the properties of the graphical calculus, it is easy to deduce the following symmetry property:

$$\| \begin{bmatrix} \alpha & \beta & \mu \\ \gamma & \lambda & \nu \end{bmatrix} \|_{\mathrm{HS}} = \sqrt{\frac{\dim[\mu] \dim[\nu]}{\dim[\beta] \dim[\lambda]}} \| \begin{bmatrix} \alpha & \mu & \beta \\ \gamma & \nu & \lambda \end{bmatrix} \|_{\mathrm{HS}}. \qquad (7)$$

That is, by swapping two columns (corresponding to reflecting the diagram about the α-axis), we pick up a dimension factor according to the corresponding irreducible representations. We remark that this relation has a well-known counterpart for SU(2): there, the Clebsch–Gordan series is multiplicity-free and (7) holds for the absolute values.

Therefore, given a tripartite quantum state ρ_{ABC}, Theorem 1, the symmetry relation (7) and a polynomial upper bound imply that

$$\frac{\dim[\mu] \dim[\nu]}{\dim[\beta] \dim[\lambda]} \geq \frac{1}{\mathrm{poly}(k)} \qquad (8)$$

for a sequence of normalised Young diagrams converging to the respective spectra for the reduced density matrices. Since for large k, $\frac{1}{k} \log_2 \dim[\lambda] \to H(r) = \sum_i -\bar{\lambda}_i \log_2 \bar{\lambda}_i$, we conclude that the von Neumann entropy is strongly subadditive [32],

$$H(\rho_{AB}) + H(\rho_{BC}) \geq H(\rho_B) + H(\rho_{ABC}) \qquad (9)$$

for all quantum states ρ_{ABC}. Weak monotonicity, $H(\rho_{AB}) + H(\rho_{BC}) \geq H(\rho_A) + H(\rho_C)$, follows similarly by swapping the columns $(\alpha, \gamma) \leftrightarrow (\mu, \nu)$ instead of $(\beta, \lambda) \leftrightarrow (\mu, \nu)$ in (7).

In conclusion, we have shown that the existence of tripartite quantum states with certain marginal eigenvalues determines the asymptotic behavior of the recoupling coefficients of the symmetric group. Our methods directly generalise to higher recoupling coefficients (the analogs of general Wigner $3nj$-symbols) and quantum states of several particles: just as Theorem 1 characterizes six of the seven marginal spectra (with r_{AC} missing), in general a linear number of marginal spectra can be fixed—suggesting that higher-order representation-theoretic structures might play a role in the characterization of all marginal spectra. In this sense, our result may be regarded as a partial quantum-mechanical version of Chan and Yeung's description of the set of local Shannon entropies in terms of sizes of Young subgroups [33]. We hope that our work might provide some useful perspective in the search for new entropy inequalities for the von Neumann entropy, the "laws of quantum information theory" [34–36].

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