Relative entropy and squashed entanglement

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Entanglement and entanglement measures

- Entanglement is an important concept and resource. It is the most outstanding non-classical feature of compound states that can't be expressed as mixture of product states.

- Entanglement measures: To understand entanglement, we need entanglement measures with good properties.

  - operational ones: E_c, E_d, K_d ...
  - abstract ones: E_f, E_r, E_{sq}, E_n ...

Squashed entanglement and relative entropy of entanglement

- Squashed entanglement (Tucci '99, '02; Christandl & Winter '04)

$$E_{sq}(\rho_{AB}) := \inf \left\{ \frac{1}{2} I(A;B|E)_\rho : \rho_{ABE} \text{ is an extension of } \rho_{AB} \right\},$$

where $I(A;B|E)_\rho$ is the quantum conditional mutual information of ρ_{ABE},

$$I(A;B|E)_\rho := S(\rho_{AE}) + S(\rho_{BE}) - S(\rho_{ABE}) - S(\rho_E)$$

- It has many nice properties (monogamy, additivity, etc) and operational meaning (Koashi & Winter '04; Christandl & Winter '04 ; Devetak & Yard '08)

- Relative entropy of entanglement (Vedral et al '97, '98)

$$E_r(\rho_{AB}) := \min_{\sigma_{AB} \in \text{SEP}} D(\rho\|\sigma) \text{ with } D(\rho\|\sigma) = \text{Tr}(\rho(\log \rho - \log \sigma))$$

- Regularized version admits operational meaning. (Brandao & Plenio '08, '10)

$$E_r^\infty(\rho_{AB}) := \lim_{n\to\infty} \frac{1}{n} E_r(\rho_{AB}^{\otimes n})$$

- Post-measurement relative entropy of entanglement, with respect to restricted measurement classes (Piani '09)

$$E_{r,\mathsf{M}}(\rho_{AB}) := \inf_{\sigma \in \mathrm{SEP}} \sup_{\mathcal{M} \in \mathsf{M}} D\big(\mathcal{M}(\rho_{AB}) \| \mathcal{M}(\sigma_{AB})\big)$$

Here M is a class of measurement, such as LO, 1-LOCC, LOCC, SEP and PPT.

Outline

- Results:

  1. Monogamy relation for relative entropy of entanglement

  2. Commensurate lower bound for squashed entanglement

  3. Properties of $E_{r,\mathrm{M}}(\rho_{AB})$: asymptotic continuity and evaluation on maximally entangled states and pure states.

  4. Comparisons between entanglement measures.

- Proofs (of 1 and 2)

Monogamy relation for relative entropy of entanglement

For an entanglement measure f, one would expect the monogamy

$$f(\rho_{1:23}) \geq f(\rho_{1:2}) + f(\rho_{1:3})$$

This is true for E_{sq} but fails for E_r (along with most other EMs).
Counterexample: anti-symmetric states! (Christandl, Schuch, Winter '10)

Properly weakened monogamy relation:

Theorem 1 For every tripartite quantum state ρ_{ABE}, we have

$$E_r(\rho_{B:AE}) \geq E_{r,\text{1-LOCC}}(\rho_{AB}) + E_r^{\infty}(\rho_{BE}),$$

$$E_r^{\infty}(\rho_{B:AE}) \geq E_{r,\text{1-LOCC}}^{\infty}(\rho_{AB}) + E_r^{\infty}(\rho_{BE}).$$

Commensurate lower bound for squashed entanglement

- Squashed entanglement is faithful. (Brandão, Christandl, Yard '10)

$$E_{sq}(\rho_{AB}) > 0 \quad \Leftrightarrow \quad \rho_{AB} \text{ entangled}$$

The main result of the proof is the following 1-LOCC trace-norm bound:

$$E_{sq}(\rho_{AB}) \geq \frac{1}{16 \ln 2} \min_{\sigma_{AB} \in \text{SEP}} \|\rho_{AB} - \sigma_{AB}\|^2_{\text{1-LOCC}}$$

where

$$\|\rho_{AB} - \sigma_{AB}\|_{\text{1-LOCC}} := \sup_{\mathcal{M} \in \text{1-LOCC}} \|\mathcal{M}(\rho_{AB}) - \mathcal{M}(\sigma_{AB})\|$$

Commensurate lower bound for squashed entanglement

- We provide a 1-LOCC relative entropy lower bound for E_{sq}

Theorem 2 For every quantum state ρ_{AB}, we have

$$E_{sq}(\rho_{AB}) \geq \frac{1}{2} E^{\infty}_{r,\text{1-LOCC}}(\rho_{AB}) \geq \frac{1}{2} E_{r,\text{1-LOCC}}(\rho_{AB}).$$

- Strong subadditivity: $S(\rho_{AE}) + S(\rho_{BE}) - S(\rho_{ABE}) - S(\rho_E) \geq 0$ (Lieb, Ruskai '73)

Corollary (Refinement of strong subadditivity):

$$S(\rho_{AE}) + S(\rho_{BE}) - S(\rho_{ABE}) - S(\rho_E) \geq E_{r,\text{1-LOCC}}(\rho_{AB}).$$

(note: $E_{r,\text{1-LOCC}}(\rho_{AB}) := \inf_{\sigma \in \text{SEP}} \sup_{\mathcal{M} \in \text{1-LOCC}} D(\mathcal{M}(\rho_{AB}) \| \mathcal{M}(\sigma_{AB}))$)

Commensurate lower bound for squashed entanglement

About the new bound:

- Recovering the 1-LOCC trace-norm bound:

  applying Pinsker's inequality $D(\rho\|\sigma) \geq \frac{1}{2\ln 2}\|\rho-\sigma\|_1^2$,

  we are able to recover the 1-LOCC trace-norm bound

  (with slightly better constant factor) :

$$E_{sq}(\rho_{AB}) \geq \frac{1}{4\ln 2} \min_{\sigma_{AB}\in\text{SEP}} \|\rho_{AB}-\sigma_{AB}\|_{\text{1-LOCC}}^2$$

- It is asymptotically normalized: for maximally entangled state Φ_d and pure state ψ_{AB} ,

$$E_{r,\text{1-LOCC}}(\Phi_d) = \log(d+1) - 1$$

$$E_{r,\text{1-LOCC}}^{\infty}(\psi_{AB}) = S(\mathrm{Tr}_B\,\psi)$$

Properties of $E_{r,\mathrm{M}}(\rho_{AB})$

- Asymptotic continuity

Proposition 3 Let ρ, ρ' be two states of dimension k, with $\|\rho - \rho'\|_{\mathrm{M}} \leq \epsilon \leq \frac{1}{e}$. Then

$$\left|E_{r,\mathrm{M}}(\rho) - E_{r,\mathrm{M}}(\rho')\right| \leq 2\epsilon \log \frac{6k}{\epsilon}.$$

- Evaluation on maximally entangled states and pure states

Proposition 4 For rank-d maximally entangled state Φ_d and pure state ψ_{AB},

$$E_{r,\mathrm{LO}}(\Phi_d) = E_{r,\text{1-LOCC}}(\Phi_d) = E_{r,\mathrm{LOCC}}(\Phi_d) = E_{r,\mathrm{SEP}}(\Phi_d) = E_{r,\mathrm{PPT}}(\Phi_d) = \log(d+1) - 1,$$

$$E^{\infty}_{r,\mathrm{LO}}(\psi_{AB}) = E^{\infty}_{r,\text{1-LOCC}}(\psi_{AB}) = E^{\infty}_{r,\mathrm{LOCC}}(\psi_{AB}) = E^{\infty}_{r,\mathrm{SEP}}(\psi_{AB}) = E^{\infty}_{r,\mathrm{PPT}}(\psi_{AB}) = S(\mathrm{Tr}_B\, \psi).$$

Comparisons between entanglement measures

We are mainly interested in two families of entanglement measures:

- Squashed-like measures $\{ E_{sq}, \quad E_I, \quad E_{sq,c} \}$
- E_r families $\{ E_{r,\rightarrow}, \quad E_{r,\leftrightarrow}, \quad E_r \}$

- Conditional entanglement of mutual information (Yang, Horodecki, Wang '08)

$E_I(\rho_{AB}) := \frac{1}{2} \inf\{I(AA';BB')_\rho - I(A';B')_\rho\}$ with $\rho_{AA'BB'}$ being an extension of ρ_{AB}

- C-squashed entanglement (Yang at el '07)

$E_{sq,c}(\rho_{AB}) := \frac{1}{2} \inf\{I(A;B|E)_\rho\}$, where extension state $\rho_{ABE} = \sum_i p_i \rho^i_{AB} \otimes |i\rangle\langle i|_E$

- Relatives of relative entropy of entanglement (Piani '09)

$$E_{r,\rightarrow}(\rho_{AB}) := \sup_{\Lambda \in \text{LOCC}} E_{r,\text{1-LOCC}}(\Lambda(\rho_{AB})) \;,\; E_{r,\leftrightarrow}(\rho_{AB}) := E_{r,\text{LOCC}}(\rho_{AB})$$

Comparisons between entanglement measures

(focusing mainly on regularized versions)

We obtain:

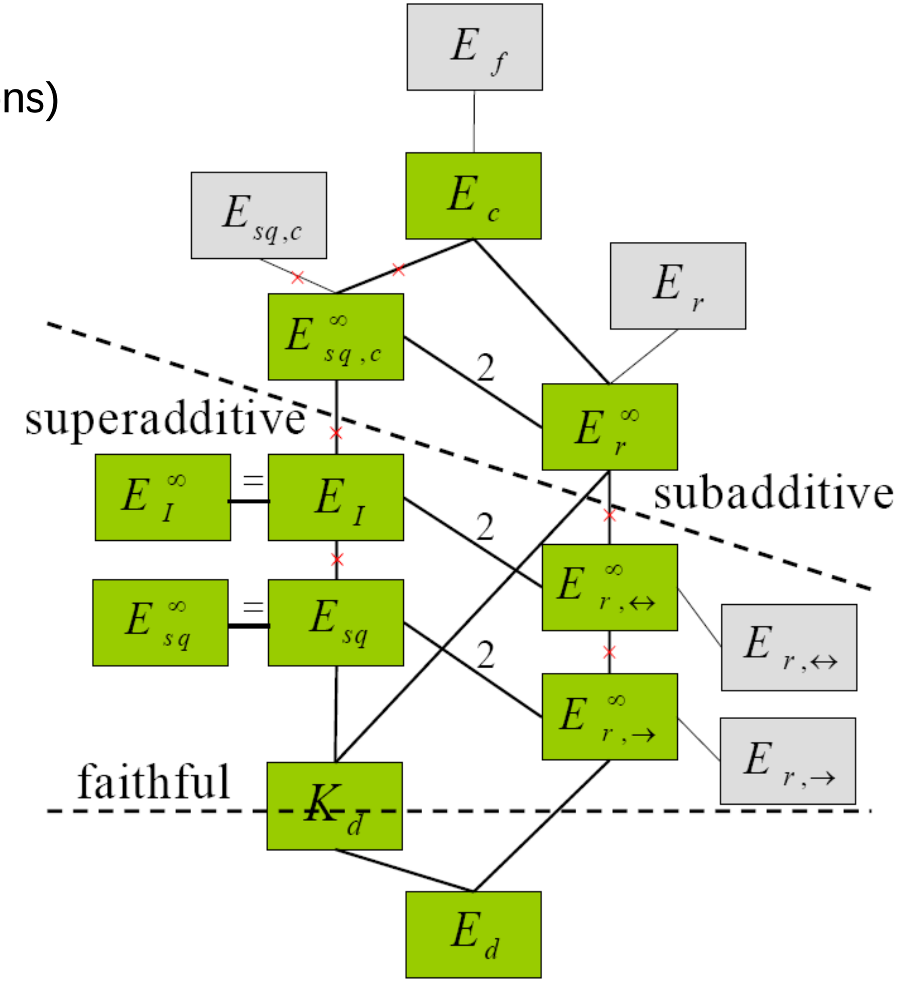
1. $2E^{\infty}_{sq,c} \geq E^{\infty}_{r}$,
2. $2E_I \geq E^{\infty}_{r,\leftrightarrow}$,
3. $2E_{sq} \geq E^{\infty}_{r,\rightarrow}$,
4. $E^{\infty}_{r,\rightarrow} \geq E_d$.

Previously known ones:

$$E_d \leq K_d \leq E_{sq} \leq E_I \leq E^{\infty}_{sq,c} \leq E_c,$$

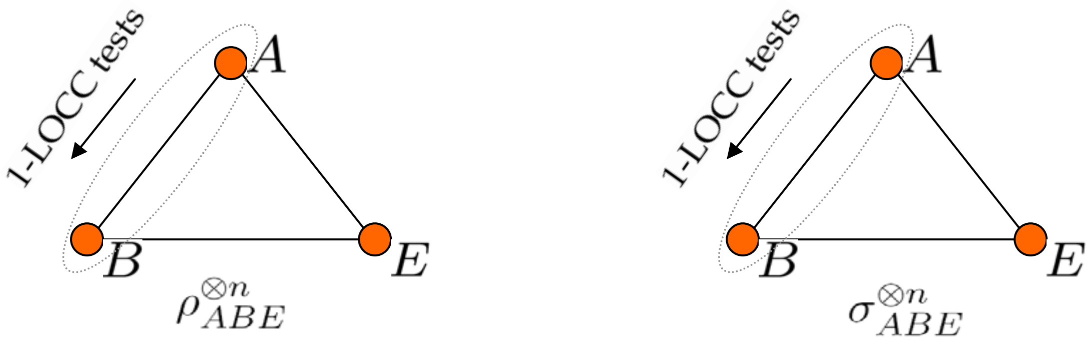
$$E^{\infty}_{r,\rightarrow} \leq E^{\infty}_{r,\leftrightarrow} \leq E^{\infty}_{r},$$

$$E_d \leq K_d \leq E^{\infty}_{r} \leq E_c.$$



Proofs of Theorem 1 and Theorem 2

Quantum hypothesis testing with one-way LOCC operations



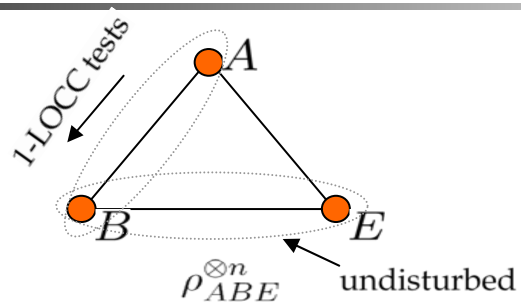
Consider the setting of hypothesis testing

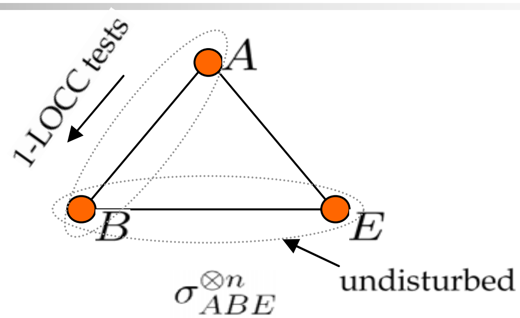
two hypotheses: $\rho_{ABE}^{\otimes n}$ or $\sigma_{ABE}^{\otimes n}$

tests allowed: $\{L_n, \mathbb{1} - L_n\}$ on $A^n B^n$; 1-LOCC implementable

two errors: $\alpha_n(L_n) := \mathrm{Tr}(\rho^{\otimes n}(\mathbb{1} - L_n))$, $\beta_n(L_n) := \mathrm{Tr}(\sigma^{\otimes n} L_n)$

Quantum hypothesis testing with one-way LOCC operations





- For any 1-LOCC $\mathcal{M}^{AB\to X}$, there exist 1-LOCC tests $\{L_n, \mathbb{1} - L_n\}$ such that

$$\lim_{n\to\infty} \alpha_n(L_n) = 0,$$

$$\lim_{n\to\infty} \frac{-1}{n} \log \beta_n(L_n) = D(\mathcal{M}(\rho) \| \mathcal{M}(\sigma))$$

Quantum Stein's Lemma
(Hiai, Petz '91)

- Meanwhile, the states $\rho_{BE}^{\otimes n}$ and $\sigma_{BE}^{\otimes n}$ are kept almost undisturbed!

Gentle Measurement Lemma
(Winter '99)

Note: one-way LOCC measurement (1-LOCC) $\mathcal{M}^{AB\to X}$ can be replaced by
one side local measurement (1-LM) $\mathcal{M}^{AB\to XB}$

A technical lemma

Lemma *For any two states ρ_{ABE} and σ_{ABE}, and any one-way LOCC measurement $\mathcal{M}^{AB\to X}$ acting on system AB, with classical communication from A to B, there exists a sequence of quantum instruments $\mathcal{T}_n^{A^nB^n\to XB^n}$, which are implementable via local operations and classical communication from A^n to B^n, such that*

$$\lim_{n\to\infty} \frac{1}{n} D(\mathcal{T}_n^c(\rho_{AB}^{\otimes n}) \| \mathcal{T}_n^c(\sigma_{AB}^{\otimes n})) = D(\mathcal{M}(\rho_{AB}) \| \mathcal{M}(\sigma_{AB})),$$

$$\lim_{n\to\infty} \left\| \mathcal{T}_n^q \otimes \mathbb{1}^{E^n}(\rho_{ABE}^{\otimes n}) - \rho_{BE}^{\otimes n} \right\|_1 = 0,$$

where $\mathcal{T}_n^c := \mathrm{Tr}_{B^n} \circ \mathcal{T}_n^{A^nB^n\to XB^n}$, *and* $\mathcal{T}_n^q := \mathrm{Tr}_X \circ \mathcal{T}_n^{A^nB^n\to XB^n}$.

Note: one-way LOCC measurement (1-LOCC) $\mathcal{M}^{AB\to X}$ can be replaced by one side local measurement (1-LM) $\mathcal{M}^{AB\to XB}$

Proof of Theorem 1------
Monogamy relation for relative entropy of entanglement

To show $E_r(\rho_{B:AE}) \geq E_{r,\text{1-LOCC}}(\rho_{AB}) + E_r^{\infty}(\rho_{BE})$

Proof:

Let $\sigma_{B:AE}$ be the nearest separable state to $\rho_{B:AE}$;
Let $\mathcal{M}^{AB\to X}$ be an arbitrary one-way LOCC measurement;
Let $\mathcal{T}_n^{A^nB^n\to XB^n}$ be the quantum instruments in the lemma associated with ρ_{ABE}, σ_{ABE} and $\mathcal{M}^{AB\to X}$.

$$
\begin{aligned}
E_r(\rho_{B:AE}) &= D(\rho_{ABE}\|\sigma_{ABE}) \\
&= \frac{1}{n} D(\rho_{ABE}^{\otimes n}\|\sigma_{ABE}^{\otimes n}) \\
\text{(monotonicity)} \quad &\geq \frac{1}{n} D(\mathcal{T}_n \otimes \mathbb{1}^{E^n}(\rho_{ABE}^{\otimes n})\|\mathcal{T}_n \otimes \mathbb{1}^{E^n}(\sigma_{ABE}^{\otimes n}))
\end{aligned}
$$

Proof of Theorem 1------
Monogamy relation for relative entropy of entanglement

Write $\mathcal{T}_n \otimes \mathbb{1}^{E^n}(\rho_{ABE}^{\otimes n}) = \sum_{i_n} p_{i_n} |i_n\rangle\langle i_n|^X \otimes \rho_{B^n E^n}^{i_n}$

$\mathcal{T}_n \otimes \mathbb{1}^{E^n}(\sigma_{ABE}^{\otimes n}) = \sum_{i_n} q_{i_n} |i_n\rangle\langle i_n|^X \otimes \sigma_{B^n E^n}^{i_n}$

$$= \frac{1}{n} D(\mathcal{T}_n^c(\rho_{AB}^{\otimes n}) \| \mathcal{T}_n^c(\sigma_{AB}^{\otimes n})) + \frac{1}{n} \sum_{i_n} p_{i_n} D(\rho_{B^n E^n}^{i_n} \| \sigma_{B^n E^n}^{i_n})$$

(joint convexity)
$$\geq \frac{1}{n} D(\mathcal{T}_n^c(\rho_{AB}^{\otimes n}) \| \mathcal{T}_n^c(\sigma_{AB}^{\otimes n})) + \frac{1}{n} D(\mathcal{T}_n^q \otimes \mathbb{1}^{E^n}(\rho_{ABE}^{\otimes n}) \| \sum_{i_n} p_{i_n} \sigma_{B^n E^n}^{i_n})$$

(1-LOCC of Tn)
$$\geq \frac{1}{n} D(\mathcal{T}_n^c(\rho_{AB}^{\otimes n}) \| \mathcal{T}_n^c(\sigma_{AB}^{\otimes n})) + \frac{1}{n} E_r(\mathcal{T}_n^q \otimes \mathbb{1}^{E^n}(\rho_{ABE}^{\otimes n}))$$

(lemma & asym. cont.)
$$\stackrel{n\to\infty}{\longrightarrow} D(\mathcal{M}(\rho_{AB}) \| \mathcal{M}(\sigma_{AB})) + E_r^\infty(\rho_{BE}).$$

Since $\mathcal{M}$ is arbitrary, we have

$$\begin{aligned} E_r(\rho_{B:AE}) &\geq \sup_{\mathcal{M} \in \text{1-LOCC}} D(\mathcal{M}(\rho_{AB}) \| \mathcal{M}(\sigma_{AB})) + E_r^\infty(\rho_{BE}) \\ &\geq E_{r,\text{1-LOCC}}(\rho_{AB}) + E_r^\infty(\rho_{BE}). \end{aligned}$$

Proof of Theorem 2------

Commensurate lower bound for squashed entanglement

To show $$E_{sq}(\rho_{AB}) \geq \frac{1}{2} E^{\infty}_{r,\text{1-LOCC}}(\rho_{AB}) \geq \frac{1}{2} E_{r,\text{1-LOCC}}(\rho_{AB})$$

Proof:

(Theorem 1)

$$E_r^{\infty}(\rho_{B:AE}) \geq E^{\infty}_{r,\text{1-LOCC}}(\rho_{AB}) + E_r^{\infty}(\rho_{BE})$$

(Brandão, Christandl, Yard '11, Lemma 1)

$$I(A;B|E)_{\rho} \geq E_r^{\infty}(\rho_{B:AE}) - E_r^{\infty}(\rho_{BE})$$

$$\Rightarrow E_{sq}(\rho_{AB}) \geq \frac{1}{2} E^{\infty}_{r,\text{1-LOCC}}(\rho_{AB}) \geq \frac{1}{2} E_{r,\text{1-LOCC}}(\rho_{AB})$$

Last remark

$$E_r(\rho_{B:AE}) \geq E_{r,\text{1-LOCC}}(\rho_{AB}) + E_r^{\infty}(\rho_{BE})$$

$$E_r^{\infty}(\rho_{B:AE}) \geq E_{r,\text{1-LOCC}}^{\infty}(\rho_{AB}) + E_r^{\infty}(\rho_{BE})$$

$$E_{sq}(\rho_{AB}) \geq \frac{1}{2} E_{r,\text{1-LOCC}}^{\infty}(\rho_{AB}) \geq \frac{1}{2} E_{r,\text{1-LOCC}}(\rho_{AB})$$

1-LM

One-way LOCC measurement can be replaced by one-side local measurement, which is a measurement on system A and an identity operation on system B.

Inspired by the results of (Brandão & Harrow ’12)

Open questions

- Applications of our results?
- Faithfulness of multipartite squashed entanglement?

Thank you!