An area law and sub-exponential algorithm for 1D systems.

I. Arad, A. Kitaev, Z. Landau, U. Vazirani

A Basic Question

Lesson from Quantum Complexity Theory:

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Heuristic techniques, DMRG, have been very successful for 1D systems.

Is there a principled phenomenon behind this?

Is there a clean well defined class of quantum many body systems that we can analyze?

Gap and Area Law

The size of the **gap** between the lowest and second lowest eigenstate:

- QMA-complete require an inverse polynomial size gap.
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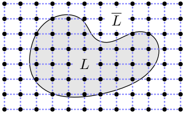
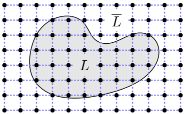
A first point of entry is a remarkable conjecture:



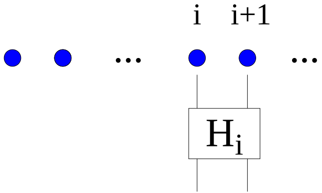
Area Law: Given a gapped local Hamiltonian, for any subset S of particles, the entanglement entropy of ρ_S, the reduced density matrix of the ground state restricted to S, is bounded by the surface area of S i.e. the number of local interactions between S and $\overline{S}$.

Basic Questions



- Can you prove an area law?
- If so, do these states have small working descriptions?
- Can they be efficiently computed?

Concretely in 1D



Given:

- n d-dimensional particles on a line, $\mathcal{H} = (\mathbb{C}^d)^{\otimes n}$,
- local operators $0 \leq H_i \leq 1$ acting non-trivially on the ith and $i+1$st particle.
- a Hamiltonian $H = \sum_i H_i$ with a gap ϵ between the energy of the ground state and the next lowest energy.

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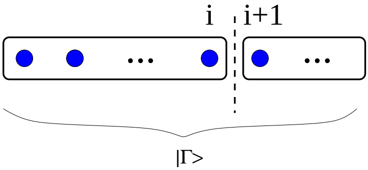

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Goal: structural properties of the ground state $|\Gamma>$.

Result: 1D Area Law

Previous results for 1D:

- Hastings (2007) with bound $e^{O(\log d/\epsilon)}$.
  - Existence of an MPS with polynomial bond dimension.
  - Finding an approximation to the ground state is $\in NP$.
- Arad, Landau, Vazirani (2011): $\tilde{O}(\frac{\log d}{\epsilon})^3$ for *frustration free* system.
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This result:

Theorem: The entanglement entropy of the ground state of a 1D gapped Hamiltonian is bounded by $\tilde{O}(\frac{\log^3 d}{\epsilon})$

- Exponential improvement of the bound.
- Bound the cusp of a 2D sub-volume law.
- Implies a sublinear bond dimension MPS which leads to . . .

Sub-exponential algorithm

Theorem: There is a subexponential time algorithm for finding an inverse polynomial approximation to ground state of a 1D gapped Hamiltonian.

Combines sublinear bond dimension with a dynamical programing algorithm (Aharonov, Arad, Irani, 2009).

Preliminaries: Entanglement Rank

For a vector $v \in \mathcal{H}_1 \otimes \mathcal{H}_2$ with Schmidt decomposition $v = \sum_{i=1}^{D} a_i \otimes b_i$, has *entanglement rank* D.

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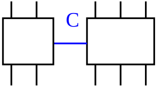

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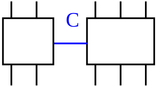
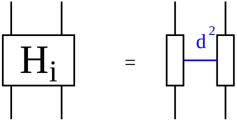


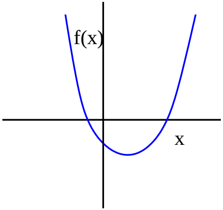
- Local operator H_i can only increase the entanglement rank across $i, i+1$ by d^2.



Preliminaries: Functional calculus of an operator

What does the operator $f(H) = H^2 - 2H - 1$ look like?

- Same eigenspaces as H,
- Eigenvalue x for H becomes eigenvalue $f(x)$ for $f(H)$.



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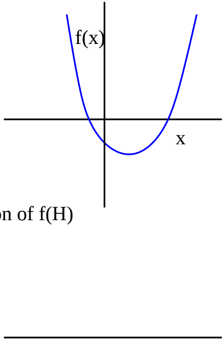

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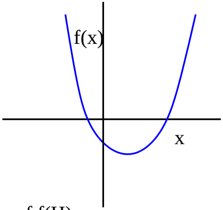

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(Located at corresponding eigenvalues)

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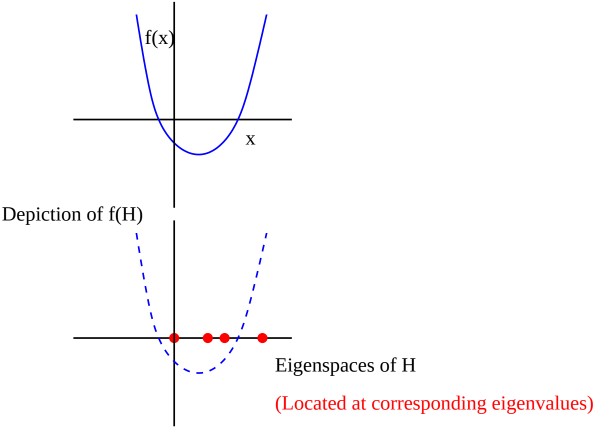

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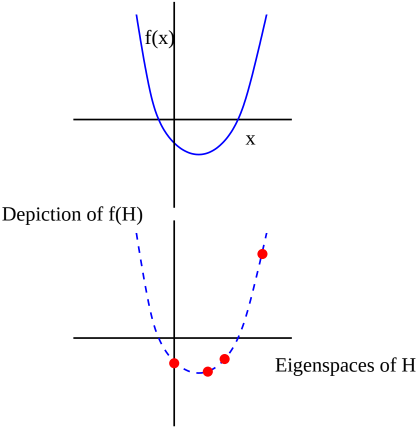

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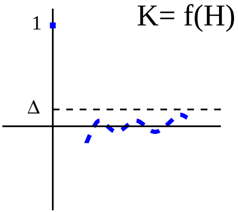

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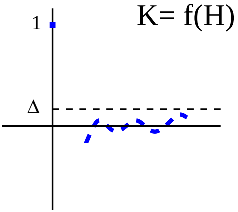
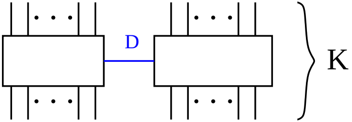


- It doesn't increase the entanglement too much:



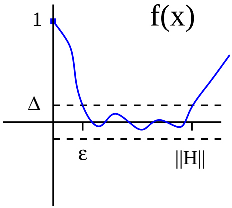
Such an operator is a (D, Δ) Approximate Ground State Projection (AGSP).

The consequence of a good AGSP: An area law

Theorem (Area Law) [Arad, Landau, Vazirani] The existence of an AGSP K for which $D\Delta < 1/2$ proves that the ground state has entropy $O(1) \log D$.

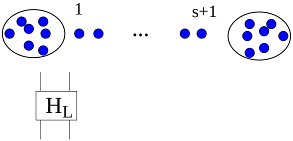
Building good AGSP's: reduce the norm

Looking for low entanglement operators that look like:



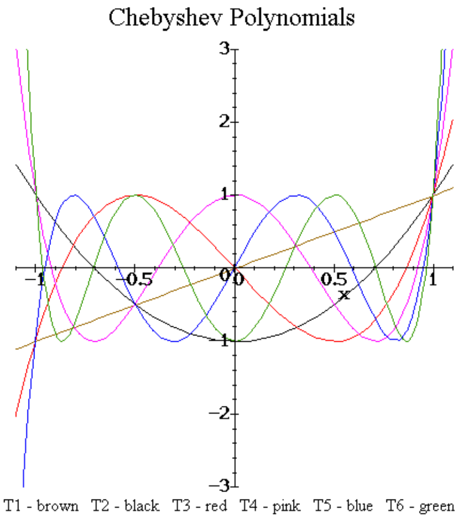
Smaller $||H||$ would be better but we don't want to lost the local structure around the cut.

Solution: Replace $H = \sum_i H_i$ with $H' = H_L + H_1 + H_2 + \cdots + H_s + H_R$.



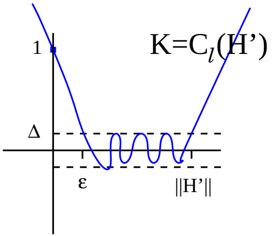
Building good AGSP's: Chebyshev polynomials

Chebyshev polynomials: small in an interval:



A good AGSP

A dilation and translation of the Chebyshev polynomial gives:



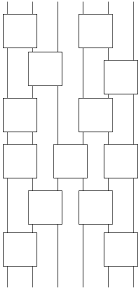
with

$$\Delta = e^{-\frac{\ell\sqrt{\epsilon}}{\sqrt{||H'||}}}.$$

Entanglement Increase due to a single term of $(H')^\ell$

$$(H')^\ell = \sum(\text{ product of } H_j).$$

For a single term:



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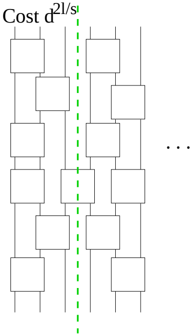
For a single term:

- Across some cut, an average number of terms are involved $\rightarrow d^{2\ell/s}$.



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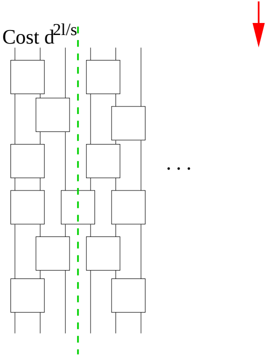

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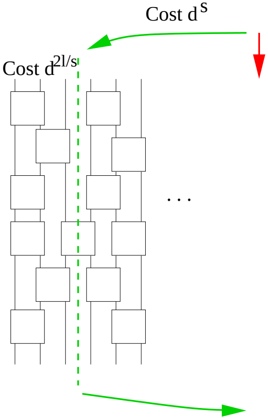

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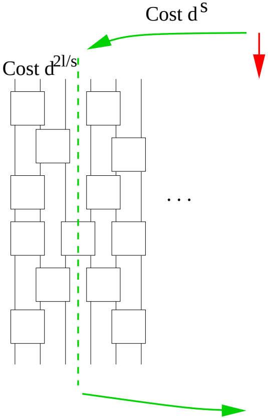
Total: $d^{2\ell/s+s}$



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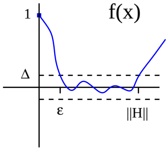
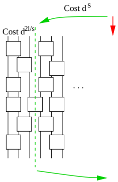
Need to group terms in a nice way but it all works out with total entanglement increase of the same order as the single term.

Putting things together: Area Law for H'

Chebyshev $C_\ell(H')$ has $\Delta \approx e^{-O(\ell\sqrt{\epsilon}/\sqrt{s})}$:



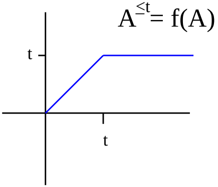
Entanglement analysis yields $D \approx O(d^{\ell/s+s})$.



Chosing $\ell = s^2$ yields $log(D\Delta) \approx -s^{3/2}\sqrt{\epsilon} + s\log d$. Approximate equality occurs with $s \approx \log^2 d/\epsilon$ which yields $D \approx \log^3 d/\epsilon$.

From H' to H: truncation and the definition of H_L and H_R

The **truncation**, $A^{\leq t}$ of an operator A:



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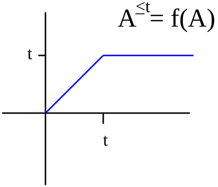
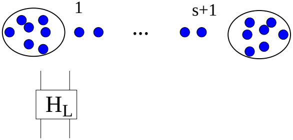


Definition of H_L and H_R using truncation:



$$H_L = (\sum_{i<1} H_i)^{\leq t}, \qquad H_R = (\sum_{i>s+1} H_i)^{\leq t}.$$

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Question How does the Hamiltonian $H' = H_L + H_1 + \cdots + H_s + H_R$ compare to $H = \sum_j H_j$?

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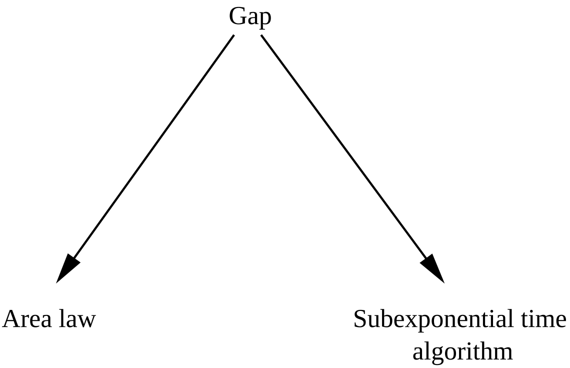
> **Robustness Theorem:** *The gaps of H and H' are of the same order and the ground states of H and H' are within $\exp(-t)$.*

Area law for H now follows by starting with a constant truncation level $t = t_0$ and then letting it grow to $t = O(\log n)$.

Summary



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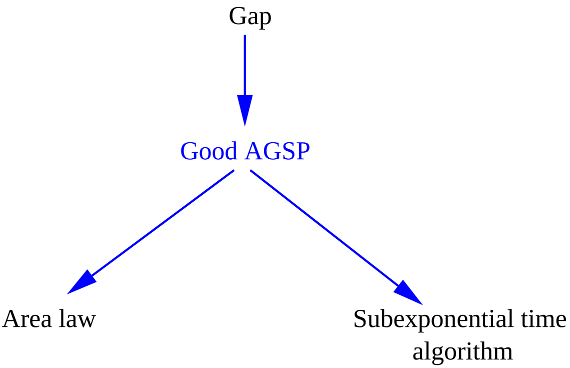

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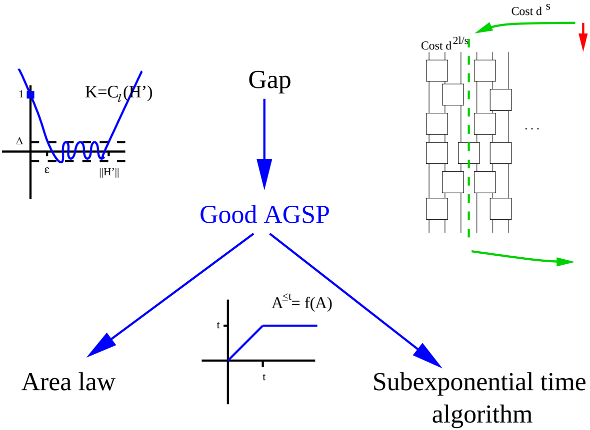

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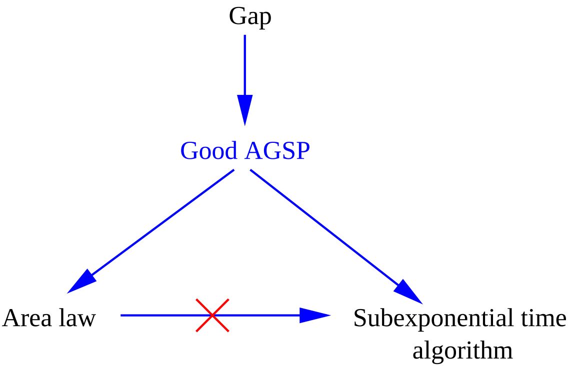

Schuch, Cirac, Verstraete

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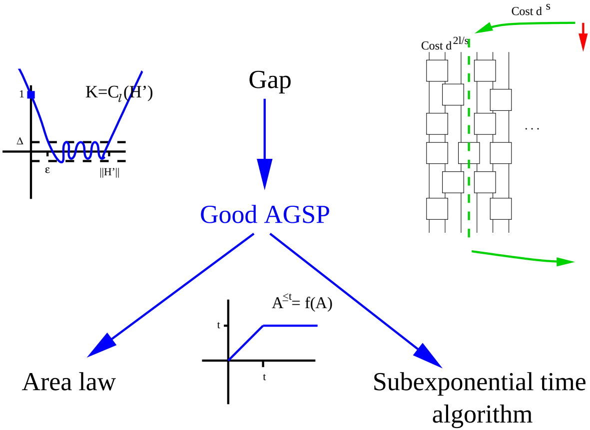

The structural engine for these results are AGSP's.

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- Of independent interest: robustness theorem of truncation.