Rank-one and Quantum XOR Games

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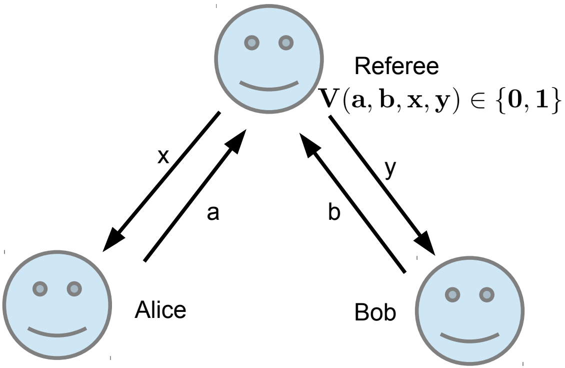
Two Player, One Round Games



- Computational Complexity
  - Interactive proof systems
  - Efficient proof verification
  - PCP theorem
  - Hardness of approximation

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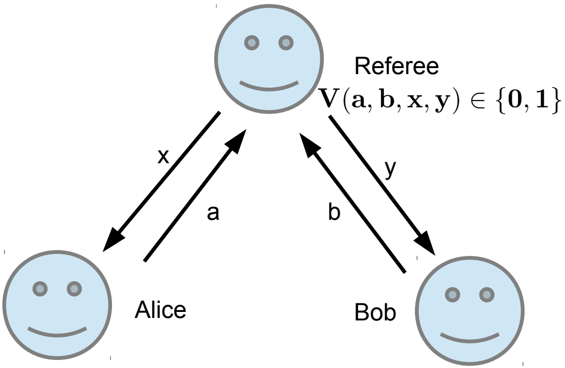
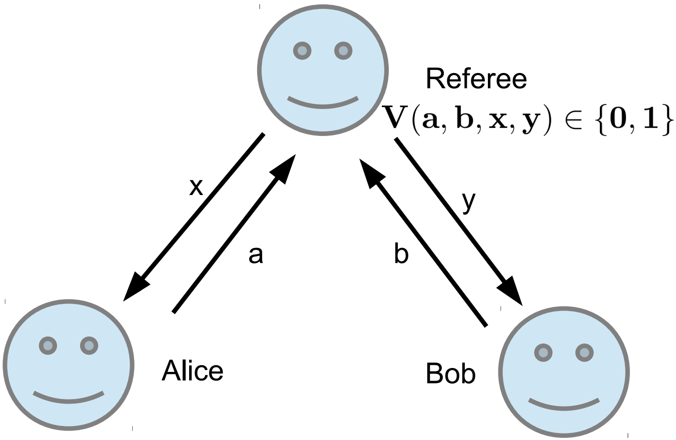


- Computational Complexity
  - Interactive proof systems
  - Efficient proof verification
  - PCP theorem
  - Hardness of approximation
- Nonlocality/Bell inequalities

Classical XOR Games



- Classical XOR games: $\{a, b\} \in \{0, 1\}$
- $V(a, b, x, y) = V(a \oplus b, x, y)$.

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- Entangled bias $\omega^{*}(G)$

Classical XOR Games versus Quantum XOR

- For all classical XOR games G, we have

$$\omega(G) \leq \omega^*(G) \leq K\omega(G),$$

where $1.67 \leq K \leq 1.783$, [CHTW04]. ✓

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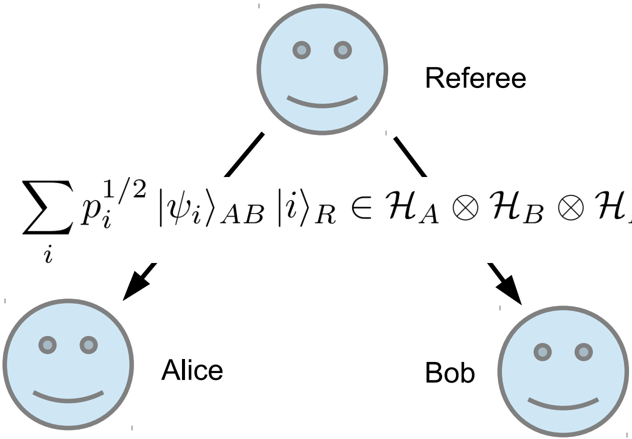
$$\omega^*(G^{\otimes n}) = \omega^*(G)^n$$
- Quantum XOR: Unbounded Violation of Perfect Parallel Repetition
$$\omega^*(C_n^{\otimes 2}) \geq \frac{n}{2}\omega^*(C_n)^2$$ ✗

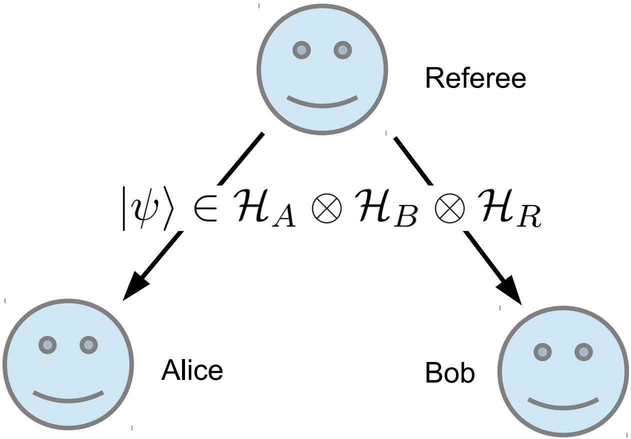
Quantum XOR Games



$$\sum_i p_i^{1/2} |\psi_i\rangle_{AB} |i\rangle_R \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_R$$

Referee prepares (known) state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_R$ and sends register A to Alice, B to Bob.
Referee has private register $\mathcal{H}_R$.

Quantum XOR Games



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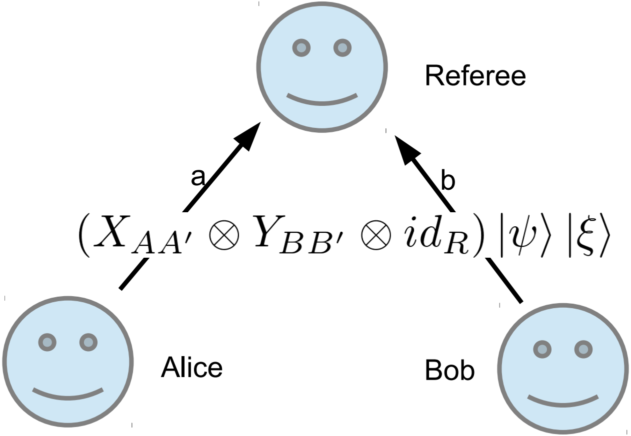
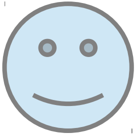


Alice and Bob share an entangled state $|\xi\rangle \in \mathcal{H}_{A'} \otimes \mathcal{H}_{B'}$.
Alice and Bob apply ± 1-*observables* $X_{AA'} = X^0 - X^1$, $Y_{BB'} = Y^0 - Y^1$.
Return outcomes $a, b \in \{0, 1\}$ to Referee.

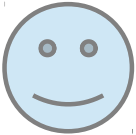
Quantum XOR Games



Referee

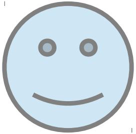
$$\text{If } a \oplus b = 0,\ \{\Pi_0^{ACC}, id_R - \Pi_0^{ACC}\}$$

$$\text{If } a \oplus b = 1,\ \{\Pi_1^{ACC}, id_R - \Pi_1^{ACC}\}$$



Alice

Bob



Referee measures private register, depending on parity of Alice and Bob's responses.

Example: T_n

Let $|\psi_n\rangle$ be the maximally entangled state in n dimensions.

T_n: Alice and Bob sent one of

$$|\phi_0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle + |\psi_n\rangle\right)$$
$$|\phi_1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle - |\psi_n\rangle\right)$$

with equal probability.
If $|\phi_0\rangle$, respond with answers of even parity.
If $|\phi_1\rangle$, respond with answers of odd parity.

Orthogonal ✓ Locally distinguishable ?

Unbounded advantage of $\omega^*(G)$ over $\omega^{me}(G)$ and $\omega(G)$

$$\omega(T_n) = \omega^{me}(T_n) = \frac{1}{\sqrt{n}}$$

$$\omega^*(T_n) = 1$$

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$\omega^*(T_n) = 1$ can only be achieved in limit of infinite entanglement.
($T_2 \longleftrightarrow$ LTW's coherent state exchange game.)

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Classical XOR games: maximally entangled states are optimal resource.
Entanglement provides advantage of at most small constant multiplicative factor: Grothendieck/Tsirelson.

Example: C_n

Referee chooses $k \in \{1, \ldots, n\}$ randomly.
Sends one of the two states

$$|\phi_{0k}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|k\rangle + |k\rangle|0\rangle\right)$$
$$|\phi_{1k}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|k\rangle - |k\rangle|0\rangle\right)$$

each chosen with probability $\frac{1}{2}$ to Alice and Bob.

If $|\phi_{0k}\rangle$, respond with answers of even parity.
If $|\phi_{1k}\rangle$, respond with answers of odd parity.

Large Violation of Perfect Parallel Repetition

- Suppose Alice and Bob play two games simultaneously and must win both "sub-games" in order to win.

  For classical XOR games, we have

$$\omega^*(G \otimes G) = \omega^*(G)^2.$$

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- Suppose Alice and Bob play two games simultaneously and must win both "sub-games" in order to win.

  For classical XOR games, we have

$$\omega^*(G \otimes G) = \omega^*(G)^2.$$

- However for Rank-one Quantum & Quantum XOR games:

$$\omega^*(C_n) = \frac{1}{n}$$

$$\omega^*(C_n \otimes C_n) \geq \frac{1}{2n} \gg (\omega^*(C_n))^2$$

Algorithms

Theorem

There exists a polynomial-time algorithm which, given as input an explicit description of a quantum XOR game G, outputs two numbers $\omega^{nc}(G)$ and $\omega^{os}(G)$ such that

$$\omega(G) \leq \omega^{me}(G) \leq \omega^{nc}(G) \leq 2\sqrt{2}\omega(G),$$
$$\omega^*(G) \leq \omega^{os}(G) \leq 2\omega^*(G).$$

Techniques

Theorem (Grothendieck's Inequality)

Suppose that s_i *and* t_j *are real numbers such that* $|s_i|, |t_j| \leq 1$*.*

Suppose that a_{ij} *are real numbers such that* $\left|\sum_{i,j} a_{ij} s_i t_j\right| \leq 1$*. Then*

$\left|\sum_{ij} a_{ij} \langle \xi_i \mid \eta_j \rangle\right| \leq k$*, for all vectors* ξ_i, η_j *in the unit ball of a real Hilbert space* $\mathcal{H}$*. It is known that* $1.67 \leq k \leq 1.782$*.*

From this it follows that for a classical XOR game,

$$\omega^*(G) \leq k\omega(G).$$

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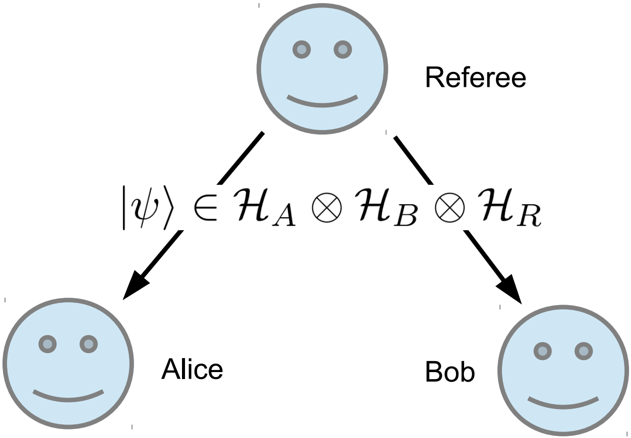
Suppose that s_i and t_j are real numbers such that $|s_i|, |t_j| \leq 1$. Suppose that a_{ij} are real numbers such that $\left|\sum_{i,j} a_{ij} s_i t_j\right| \leq 1$. Then $\left|\sum_{ij} a_{ij} \langle \xi_i \mid \eta_j \rangle\right| \leq k$, for all vectors ξ_i, η_j in the unit ball of a real Hilbert space $\mathcal{H}$. It is known that $1.67 \leq k \leq 1.782$.

Noncommutative and Operator-space extensions of Grothendieck's inequality allow us to relate biases of Quantum XOR games to SDP's.

Quantum Games



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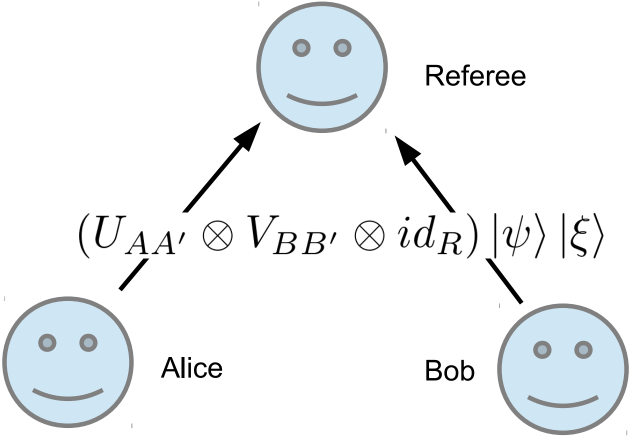
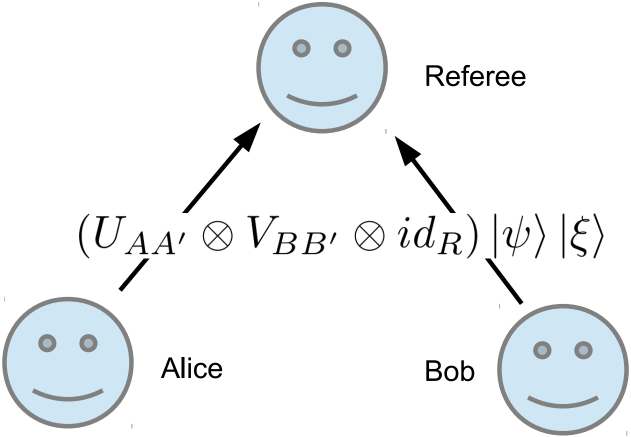


Alice and Bob share an entangled state $|\xi\rangle \in \mathcal{H}_{A'} \otimes \mathcal{H}_{B'}$.
Alice and Bob apply arbitrary local unitaries $U_{AA'}$, $V_{BB'}$ and then send registers A and B back to referee.

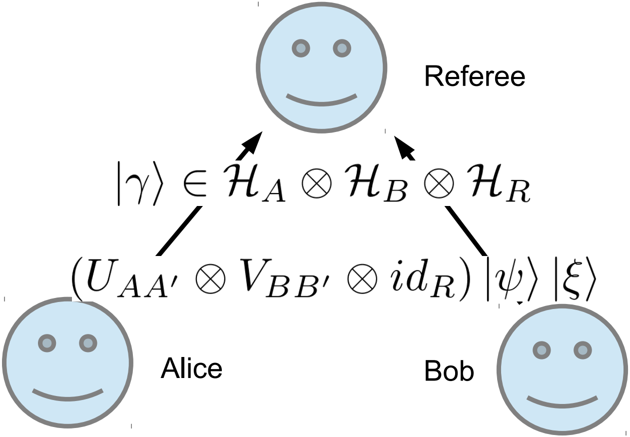
Quantum Games



Referee performs measurement with projective measurements:

$$\{P_{ACCEPT}, P_{REJECT} = Id - P_{ACCEPT}\}$$

Rank-one Quantum Games



Referee performs measurement with projective measurements:

$$\{P_{ACCEPT} = |\gamma\rangle\langle\gamma|, P_{REJECT} = Id - P_{ACCEPT}\}$$

Maximum Success Probability $= \omega_1^*(G)$

Rank-one Quantum Games $\longleftrightarrow$ Quantum XOR Games

To each Quantum XOR Game G, one can associate a Rank-one Quantum Game G' such that

$$(\omega^*(G))^2 = \omega_1^*(G)$$

To each Rank-one Quantum Game G', one can associate a Quantum XOR Game G'' such that

$$(\omega^*(G''))^2 = \omega_1^*(G')$$

Thus the previous results about SDP's and Parallel repetition can be phrased in terms of either Rank-one Quantum Games or Quantum XOR games.

Summary 1

- Classical: $\omega^*(G) \leq K\omega(G)$

  Quantum: Unbounded advantage $\omega^*(T_n) = \sqrt{n}\omega(T_n)$
- Classical: Maximally entangled state is optimal resource.

  Quantum: Unbounded advantage $\omega^*(T_n) = \sqrt{n}\omega^{me}(T_n)$
- Classical: $\omega^*(G)$ can be computed using SDP

  Quantum: $\omega^*(G)$ can be approximated up to constant factor using SDP
- Classical: Satisfies Perfect Parallel Repetition:

$$\omega^*(G^{\otimes 2}) = \omega^*(G)^2$$

  Quantum: Unbounded Violation of Perfect Parallel Repetition

$$\omega^*(C_n^{\otimes 2}) \geq \frac{n}{2}\omega^*(C_n)^2$$

Summary 2

- Generalization of classical XOR games using quantum messages.

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- Rich class of games that displays properties of entanglement not seen in classical case.
- Remain tractable with efficient approximation algorithms for biases.
- Application of deep generalizations of Grothendieck's Inequality to problems in quantum information theory.
- Operator space theory provides both examples and techniques for studying these quantum games.

Thank You!

- Rank-one Quantum Games, arXiv: 1112.3563
- Quantum XOR Games, arXiv: 1207.4939

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