Lower bounds for combinatorial polytopes, inspired by quantum communication complexity

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Joint with Samuel Fiorini (ULB), Serge Massar (ULB),

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- Yannakakis, May 2011: "I believe in fact that it should be possible to prove that there is no polynomial-size formulation for the TSP polytope or any other NP-hard problem, although of course showing this remains a challenging task"

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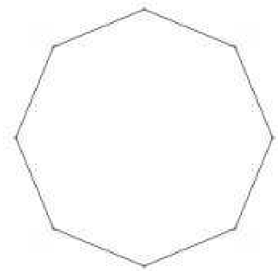

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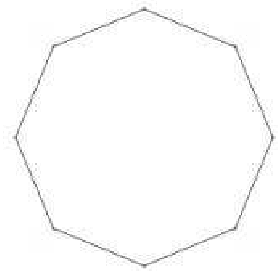

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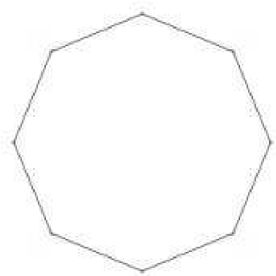

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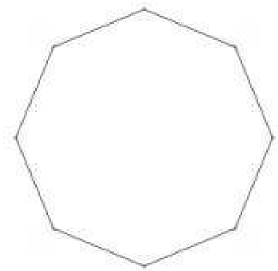

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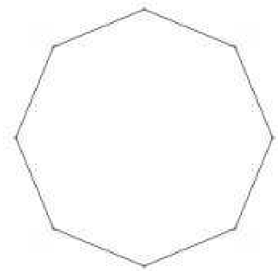

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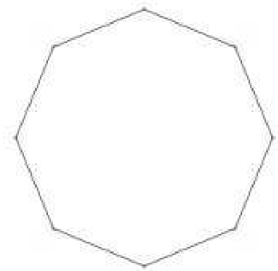

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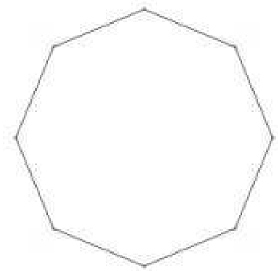

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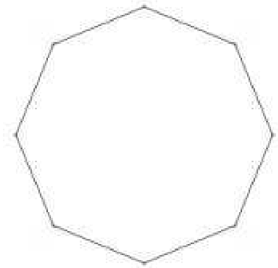

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- P_{TSP} has exponential size, so corresponding LP is huge

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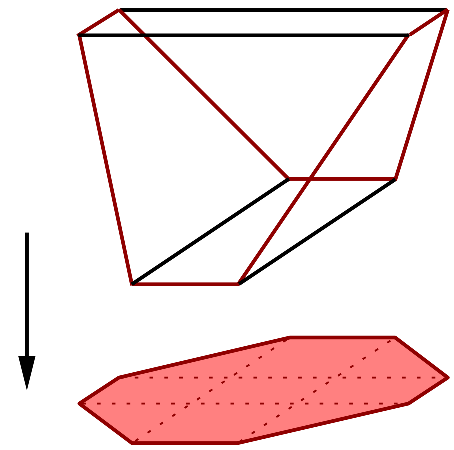

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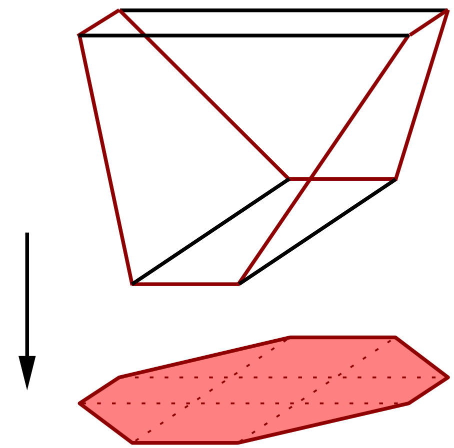

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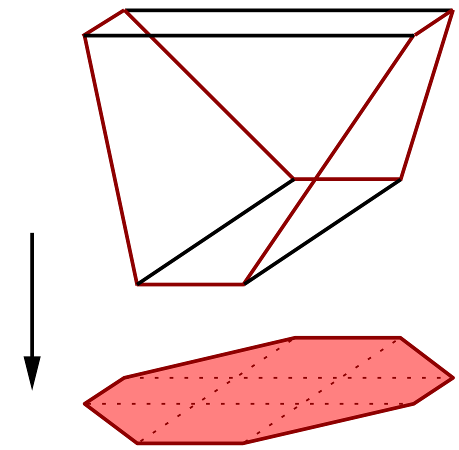

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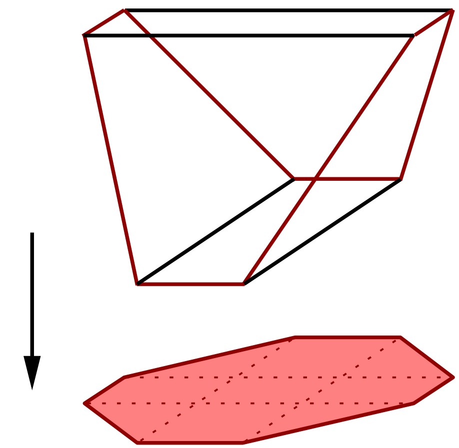

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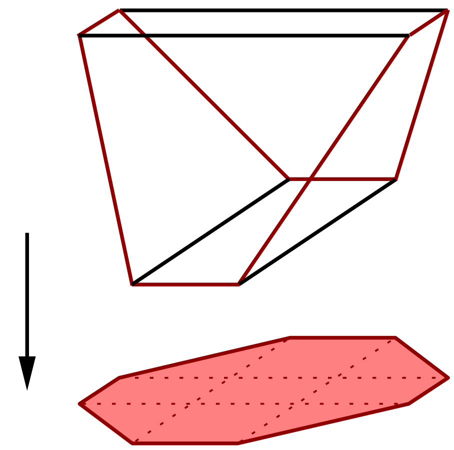

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- Our goal: strong lower bounds on $xc(P)$ for interesting P

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$\Downarrow$ gadget-based reduction

$2^{\sqrt{n}}$ lower bound for TSP-polytope

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- $\text{rank}_+(S)$ has many connections with communication complexity

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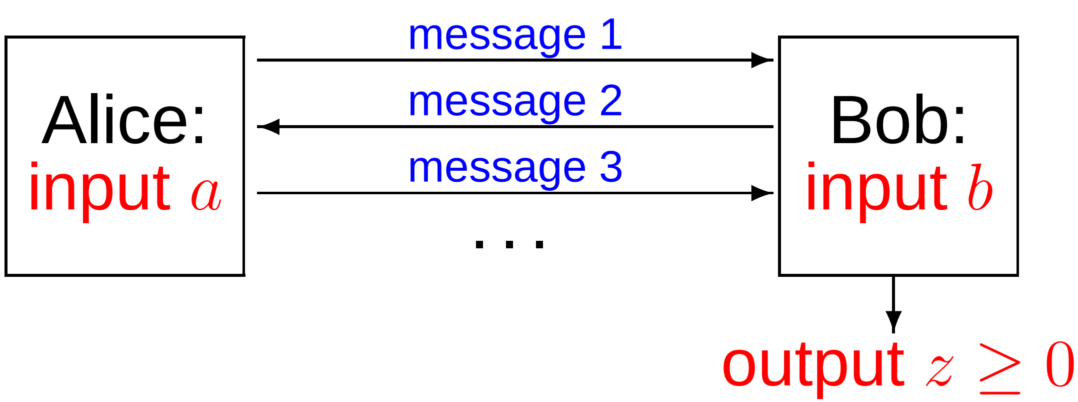

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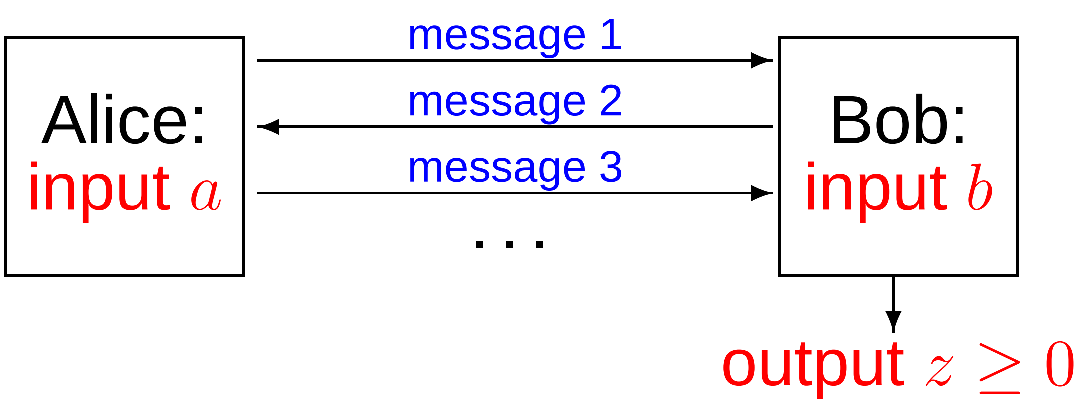

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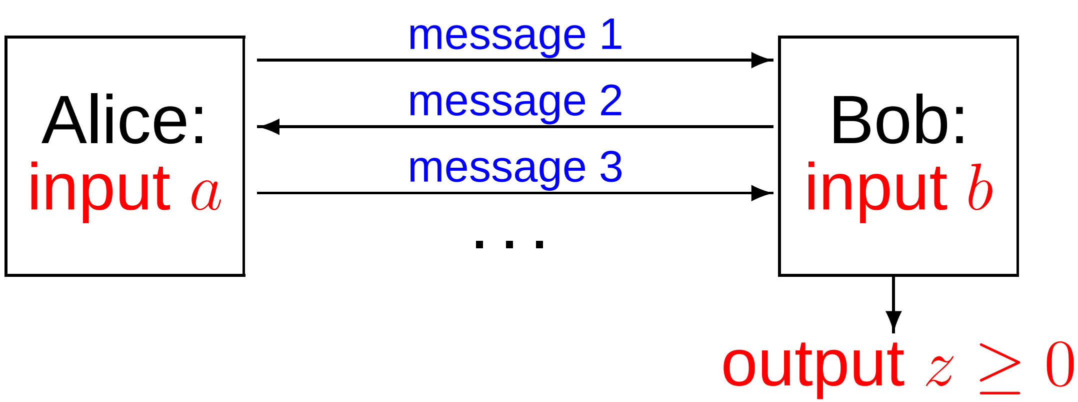

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- Can we find a matrix M where
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- There is a $O(\log n)$-qubit protocol: Alice sends $(a, 1)$, Bob measures $(b, -1)$ (ignoring normalization)

Lower bound for correlation polytope

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Cartoon by Pavel Pudlak

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