Improved learning graph based quantum algorithms for TRIANGLE and ASSOCIATIVITY

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Query complexity

Let $f : \mathcal{D} \to E$ with $\mathcal{D} \subseteq [d]^n$. Often $d = 2$ and $E = \{0, 1\}$.

Query complexity: Number of input queries needed to evaluate f.

Computational models: Deterministic, randomized, quantum

The gap between the deterministic and quantum complexities

- can be exponential [Simon'97, Shor'97]: Period finding, [EHK'99]: Hidden Subgroup Problem
- [BBCMW'01]: is at most polynomial for total functions

Quantum query complexity

Theorem [HLS'07, R'11, LMRSSz'11]: Let $f : \mathcal{D} \to \{0,1\}$ with $\mathcal{D} \subseteq [d]^n$. Then

$$
\begin{aligned}
Q(f) = \underset{u_{x,i}}{\text{minimize}} \quad & \max_{x \in \mathcal{D}} \sum_{i \in [n]} \|u_{x,i}\|^2 \\
\text{subject to} \quad & \sum_{\substack{i \in [n] \\ x_i \neq y_i}} \langle u_{x,i} | u_{y,i} \rangle = 1 \text{ for all } f(x) \neq f(y) \ ,
\end{aligned}
$$

where $u_{x,i} \in \mathbb{R}^m$, for $x \in \mathcal{D}$ and $i \in [n]$.

LEARNING GRAPHS [Belovs'11]

A learning graph $\mathcal{G}$ for $f : \mathcal{D} \to \{0,1\}$ with $\mathcal{D} \subseteq [d]^n$ is

- rooted, weighted and directed acyclic graph
- vertices labeled by $S \subseteq [n]$, the root is labeled by $\emptyset$
- An edge is $e = (S, S \cup \{i\})$ for $S \subseteq [n]$ and $i \notin S$

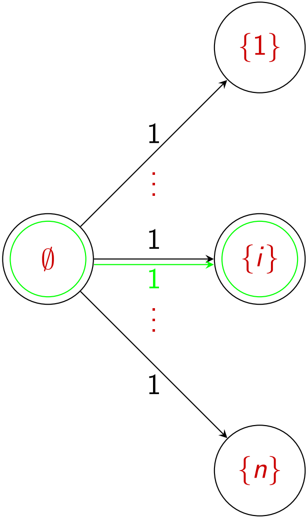
We must specify

- For every edge e its weight $w(e) \in \mathbb{R}^+$
- For every input $y \in f^{-1}(1)$ a unit flow from the root $\emptyset$, where all sinks are labeled by sets S containing a 1-certificate for y. The flow through edge e on y is denoted $p_y(e)$

We can also authorize edges $e = (S, S \cup S')$ for $S \cap S' = \emptyset$
Then by definition the length $\ell(e)$ of the edge e is $|S'|$

Learning graph for the OR function

Unit flow for the positive input $x = 0\ldots010\ldots0$, where $x_i = 1$.



Learning graphs

Learning graph complexity $\mathcal{LG}(f)$ of f

- Negative complexity of $\mathcal{G}$:

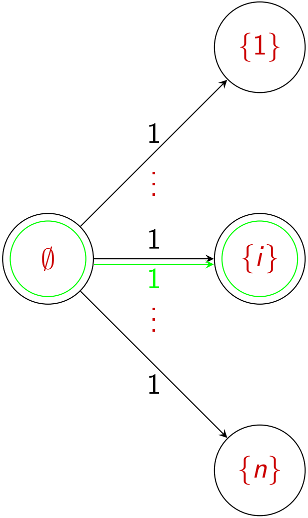
$$C_0(\mathcal{G}) = \sum_{e \in \mathcal{G}} \ell(e) w(e)$$

- Positive complexity of $\mathcal{G}$:

$$C_1(\mathcal{G}) = \max_{y \in f^{-1}(1)} \left(\sum_{e \in \mathcal{G}} \ell(e) \frac{p_y(e)^2}{w(e)} \right).$$

- Complexity of $\mathcal{G}$: $C(\mathcal{G}) = \sqrt{C_0(\mathcal{G}) C_1(\mathcal{G})}$.
- $\mathcal{LG}(f) = \min C(\mathcal{G})$ where $\mathcal{G}$ is a learning graph for f

Learning graph for the OR function



$C_0(\mathcal{G}) = n$ $\quad C_1(\mathcal{G}) = 1$ $\quad C(\mathcal{G}) = \sqrt{n}$ $\quad \mathcal{LG}(\mathrm{OR}) = O(\sqrt{n})$

Learning graph is a relaxation

Theorem[Belovs'11]: $Q(f) \leq \mathcal{LG}(f)$.

Proof Let $E_i = \{e = (S, S \cup \{i\}) : i \notin S\}$;

$$u_{x,i} = \sum_{e \in E_i} \sqrt{w(e)}|S\rangle|x_S\rangle \quad \text{for} \quad f(x) = 0$$

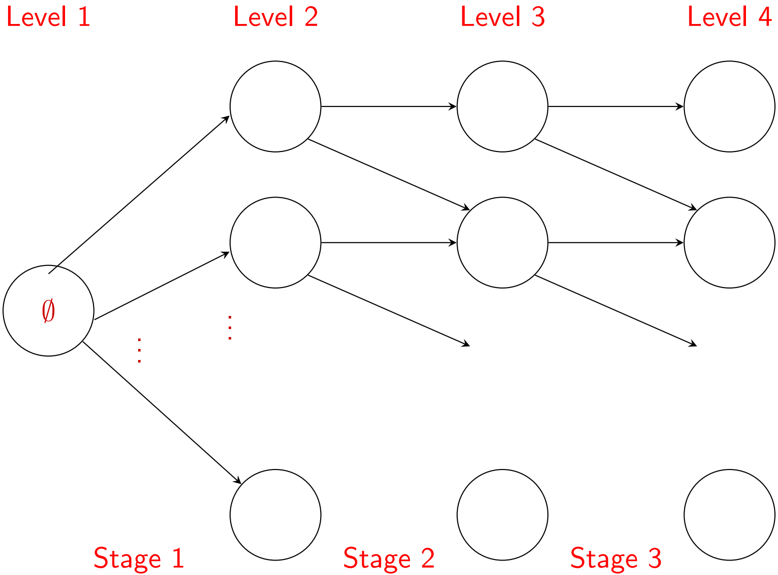
$$u_{y,i} = \sum_{e \in E_i} \frac{p_y(e)}{\sqrt{w(e)}}|S\rangle|y_S\rangle \quad \text{for} \quad f(y) = 1$$

Then $\sum_{i:x_i \neq y_i} \langle u_{x,i}|u_{y,i}\rangle$ is the flow through the cut

$$(\{S : S \subseteq I\}\,,\ \{S : I \subsetneq S\})$$

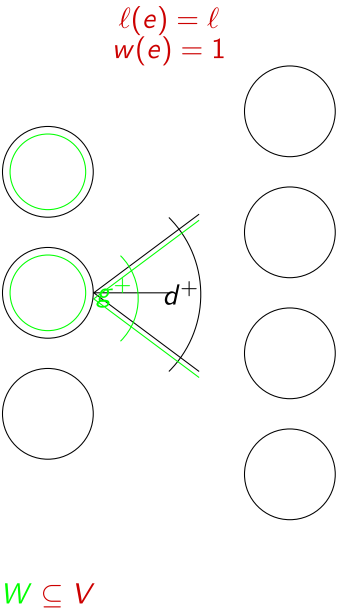
where $\quad I = \{i : x_i = y_i\}$

Learning graph in stages



Fact: Complexity of constant stages = sum of the complexities

Complexity of a stage



Complexity $= \ell \sqrt{\frac{|V|}{|W|}} \sqrt{\frac{d^+}{g^+}}$

$\frac{|V|}{|W|}$ = vertex ratio

$\frac{d+}{g+}$ = out-degree ratio

Several stages

$$C_i = \ell_i \sqrt{\frac{|V_i|}{|W_i|}} \sqrt{\frac{d_i^+}{g_i^+}}$$

The out-degre ratio is local to the stage, but the vertex ratio depends on the past

Evolution of the vertex ratio with constant in-degrees d^- and g^-:

$$|V_i| = |V_{i-1}| \frac{d_{i-1}^+}{d_i^-}; \qquad |W_i| = |W_{i-1}| \frac{g_{i-1}^+}{g_i^-}$$

$$\frac{|V_i|}{|W_i|} = \left(\frac{|V_{i-1}|}{|W_{i-1}|} \times \frac{d_{i-1}^+}{g_{i-1}^+} \right) : \frac{d_i^-}{g_i^-}$$

The in-degree ratio $\frac{d_i^-}{g_i^-}$ decreases the complexity

It depends on some well chosen database

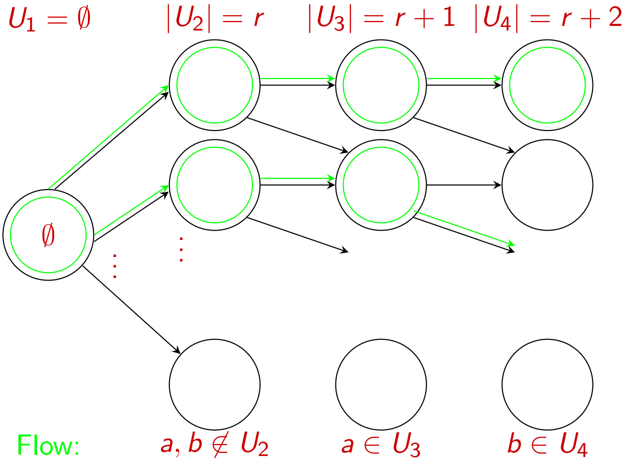
Example: Element distinctness

Element Distinctness
Oracle Input: A function $f : [n] \to [n]$.
Question: Is there a pair of distinct elements $i, j \in [n]$ such that $f(i) = f(j)$?

For every positive instance f we fix $a \neq b$ such that $f(a) = f(b)$.

Complexity of ED

$V_i = \{U_i \subseteq [n] : ...\}$



Vertex ratio at stage 3: $\frac{|V_2|}{|W_2|} = 1;\quad \frac{|V_3|}{|W_3|} = \frac{d_2^+}{g_2^+} : \frac{d_3^-}{g_3^-} = \frac{n}{1} : \frac{r}{1} = \frac{n}{r}$

Complexity: $C(\mathcal{G}) = C_1 + C_3 = r + n/\sqrt{r} = n^{2/3}$

Triangle and Subgraph$_H$

Triangle
Oracle Input: The adjacency matrix $A : \binom{n}{2} \to \{0, 1\}$ of a graph G on vertex set $[n]$.
Question: Is there a triangle in G?

Let $H = ([k], E(H))$ be some fixed k-vertex graph.

Subgraph$_H$
Oracle Input: The adjacency matrix $A : \binom{n}{2} \to \{0, 1\}$ of a graph G on vertex set $[n]$.
Question: Is there a copy of H in G?

Learning graph based algorithms

[Magniez-Santha-Szegedy'03]: $Q(\textsc{Triangle}) = O(n^{1.3})$
Database is the complete graph

[Belovs'11]: $Q(\textsc{Triangle}) = O(n^{35/27}) = O(n^{1.296})$
Sparsification: maintain just a random database where edge slots are chosen with probability $0 \leq s \leq 1$.

[Zhu'11, Lee-Magniez-Santha'11]: $Q(\textsc{Subgraph}_H) = O(n^{2-2/k-t})$, where $t = t(k, m, d) > 0$. Random database is the union of regular bipartite graphs reflecting the structure of the subgraph

[Belovs'12]: $Q(k\text{-}\textsc{Distinctness}) = O\left(n^{1-\frac{2^{k-2}}{2^k-1}}\right)$
More general learning graph: It depends also on the value of the queried variables

Our algorithms

- $Q(\textsc{Triangle}) = O(n^{9/7}) = O(n^{1.285})$
- Generalized algorithm for $\textsc{Subgraph}_H$
- $Q(\textsc{Associativity}) = O(n^{10/7}) = O(n^{1.428})$

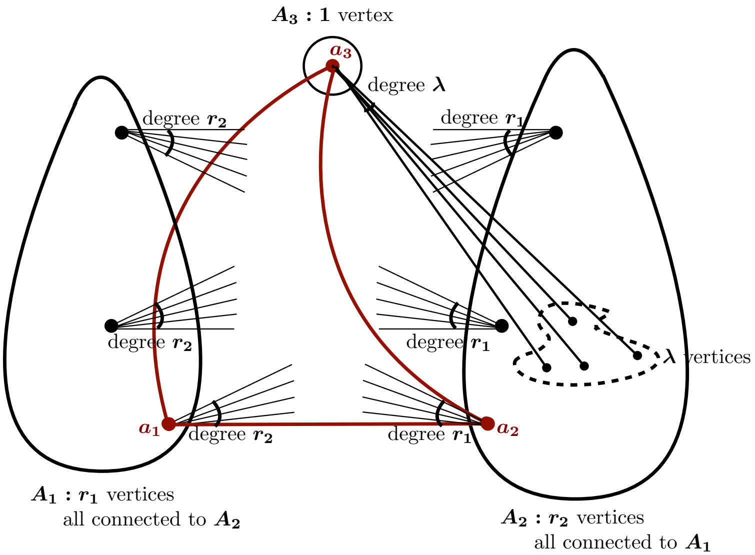
Triangle: The algorithm

For every positive instance A we fix three vertices a_1, a_2, a_3 such that they form a triangle

1. Setup: Load a complete bipartite graph between A_1 and A_2 of respective cardinality $r_1 = n^{4/7}$ and $r_2 = n^{5/7}$
2. Load a_1: Add a_1 to A_1 and connect it to all A_2
3. Load a_2: Add a_2 to A_2 and connect it to all A_1
4. Load a_3: Pick a_3 and connect it with $\lambda = n^{3/7}$ edges to A_2
5. Load $\{a_2, a_3\}$
6. Load $\{a_1, a_3\}$

Vertex sets in the bipartite graphs database can be unbalanced

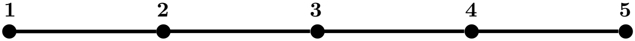
Triangle: The algorithm



Abstract language for detecting subgraphs

Let $H = ([k], E(H))$ be some fixed k-vertex graph

Example: 4-Path



Loading schedule: sequence $S = s_1 s_2 \ldots s_{k+m}$ which enumerates all vertices and edges of H.
Example: $S = [1, 2, 4, 3, (2, 1), (2, 3), (3, 4), 5, (5, 4)]$.

L-graph vertices: regular k-partite graphs with classes $A_1, \ldots, A_k$, and bipartite graphs E_{ij} between A_i and A_j for $\{i, j\} \in E(H)$.

Parameters: Set sizes $\{r_i\}$ and vertex degrees $\{d_{ij}\}$

Example: $r_1 = n, r_2 = n^{4/7}, r_3 = n^{6/7}, r_4 = n^{5/7}, r_5 = 1$;
$d_{21} = n^{6/7}, d_{23} = n^{6/7}, d_{34} = n^{5/7}, d_{54} = 1$.

Abstract language for detecting subgraphs

Theorem: There is an explicit function ϕ such that

$$\mathcal{LG}(\textsc{Subgraph}_H) \le \phi(S, \{r_i\}, \{d_{ij}\}).$$

Example: $\mathcal{LG}(4\text{-}\textsc{Path}) = O(n^{10/7})$

Best parameters can be found by linear programming:
`https://github.com/troyjlee/learning_graph_lp`

Theorem is extendable to:

- H is directed with possible self-loops
- Constant number of 1-certificates instead of just one
- Functions on labeled graphs:

  Let $f : [n]^{n\times n} \to \{0, 1\}$ be such that all minimal 1-certificate graphs are isomorphic to a fixed graph H. Then

  $$\mathcal{LG}(f) \le \mathcal{LG}(\textsc{Subgraph}_H)$$

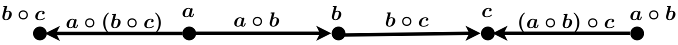
ASSOCIATIVITY

Oracle Input: Operation $\circ : [n] \times [n] \to [n]$
Question: $\exists$ a triple (a, b, c) such that $(a \circ b) \circ c \neq a \circ (b \circ c)$?

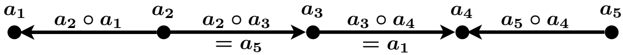
Grover search: $Q(\text{ASSOCIATIVITY}) = O(n^{3/2})$

Theorem: $Q(\text{ASSOC}) = O(n^{10/7}) = O(n^{1.428})$

Proof

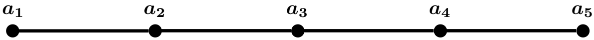


Certificate $\iff$ $a \circ (b \circ c) \neq (a \circ b) \circ c$



Certificate $\iff$ $(a_2 \circ a_3 = a_5,\ a_3 \circ a_4 = a_1$ and $a_2 \circ a_1 \neq a_5 \circ a_4)$

Certificate graph:



$Q(\text{ASSOC}) \leq \mathcal{LG}(\text{ASSOC}) \leq \mathcal{LG}(\text{4-PATH}) = O(n^{10/7}))$

Conclusion

Recent results:

- [Jeffery, Kothari, Magniez'12]: Can simulate our algorithms by quantum walks
- [Belovs, Rosmanis'12]: Our triangle algorithm is the best non-adaptive learning graph algorithm

Open problems: Complexity of

- Triangle
- Graph Collision
- k-Distinctness
- Associativity
- Matrix Product Verification