16th Workshop on Quantum Information Processing
http://conference.iiis.tsinghua.edu.cn/QIP2013

Quantum Refrigerator

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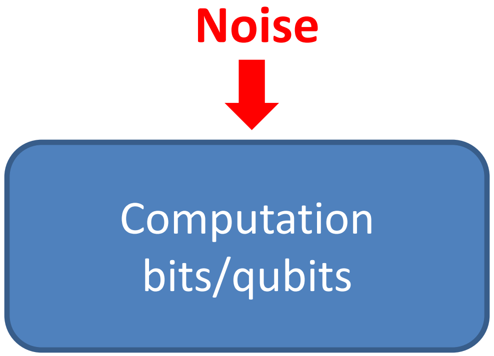
清華大學交叉信息研究院

Thermodynamics of Fault-Tolerant/Noisy Computation

Computation
bits/qubits

Thermodynamics of Fault-Tolerant/Noisy Computation



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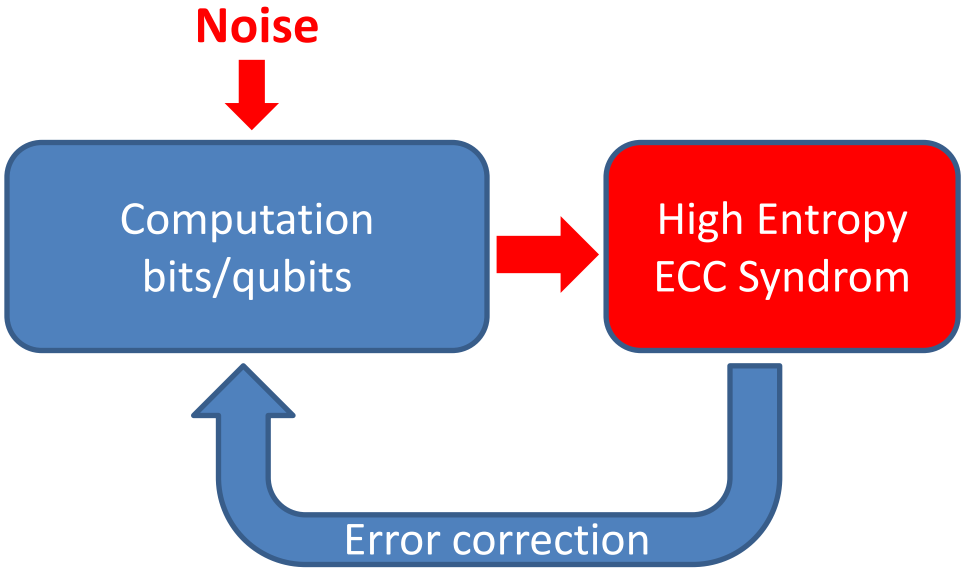

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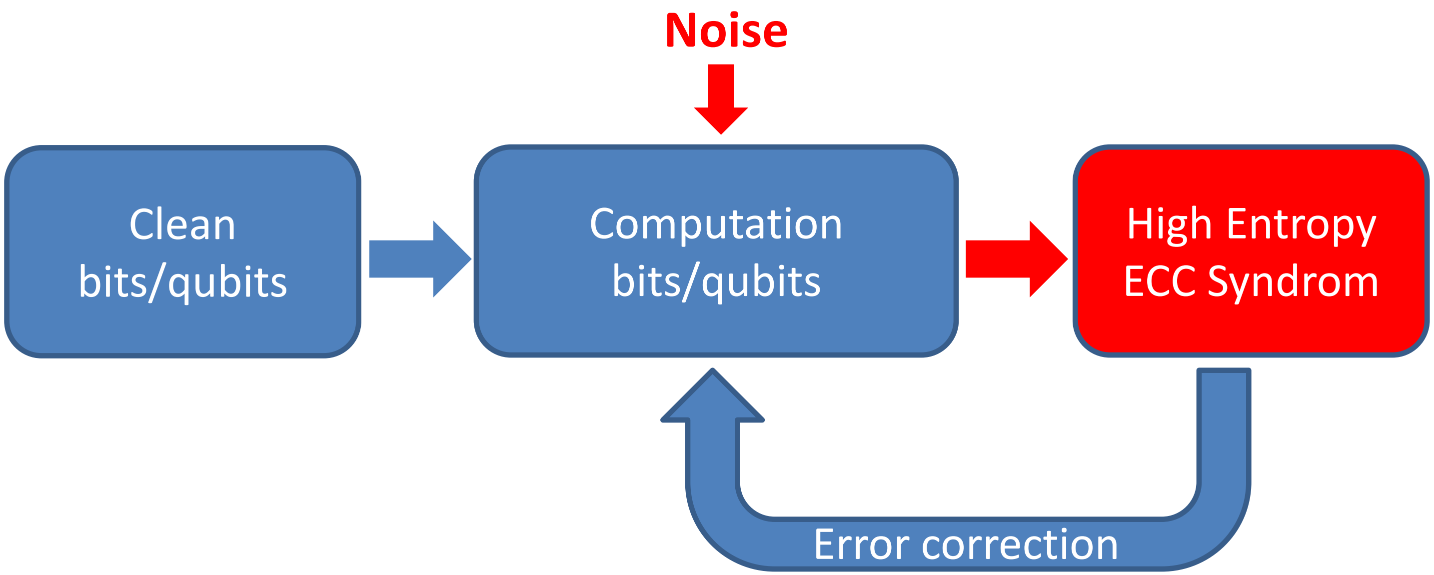

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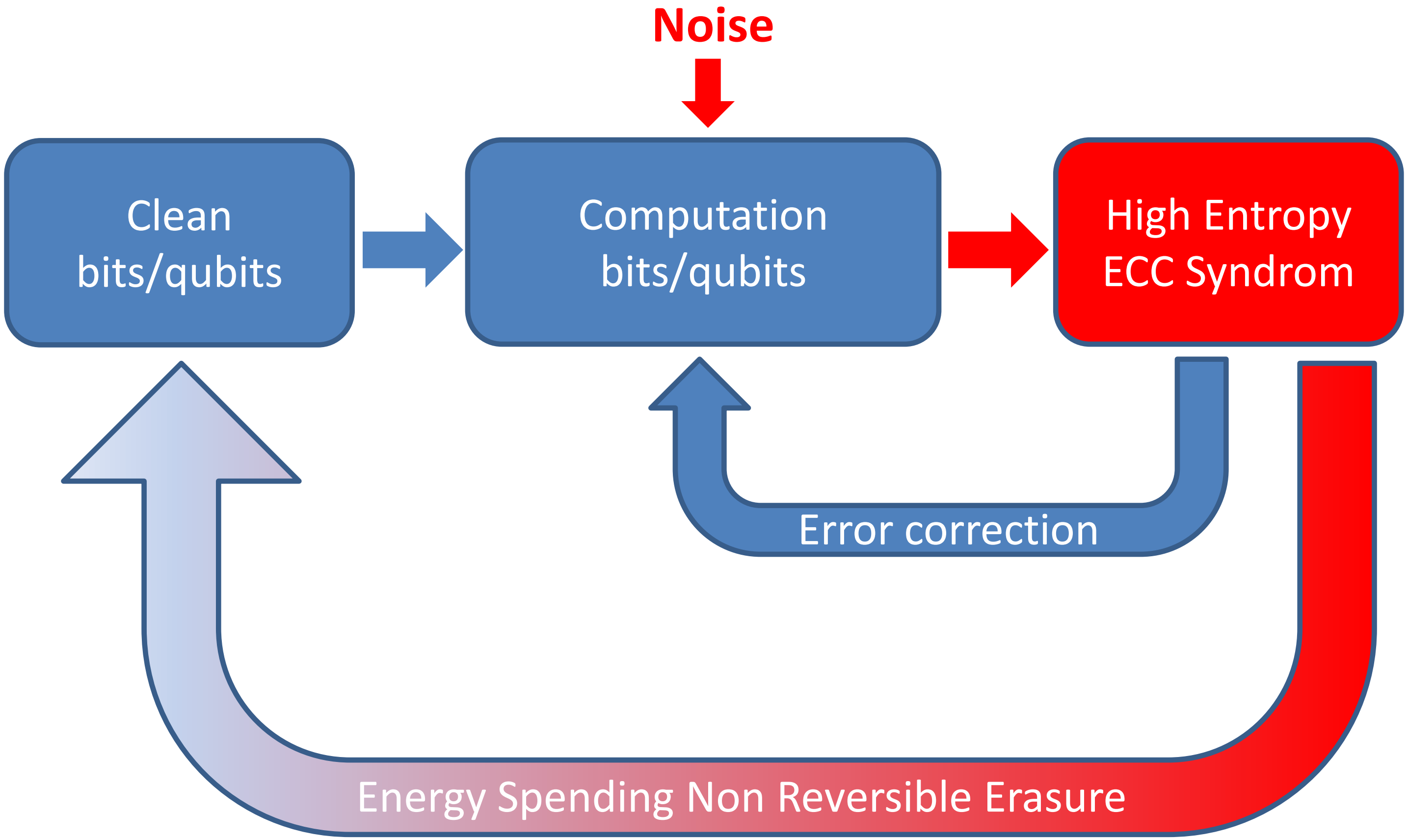

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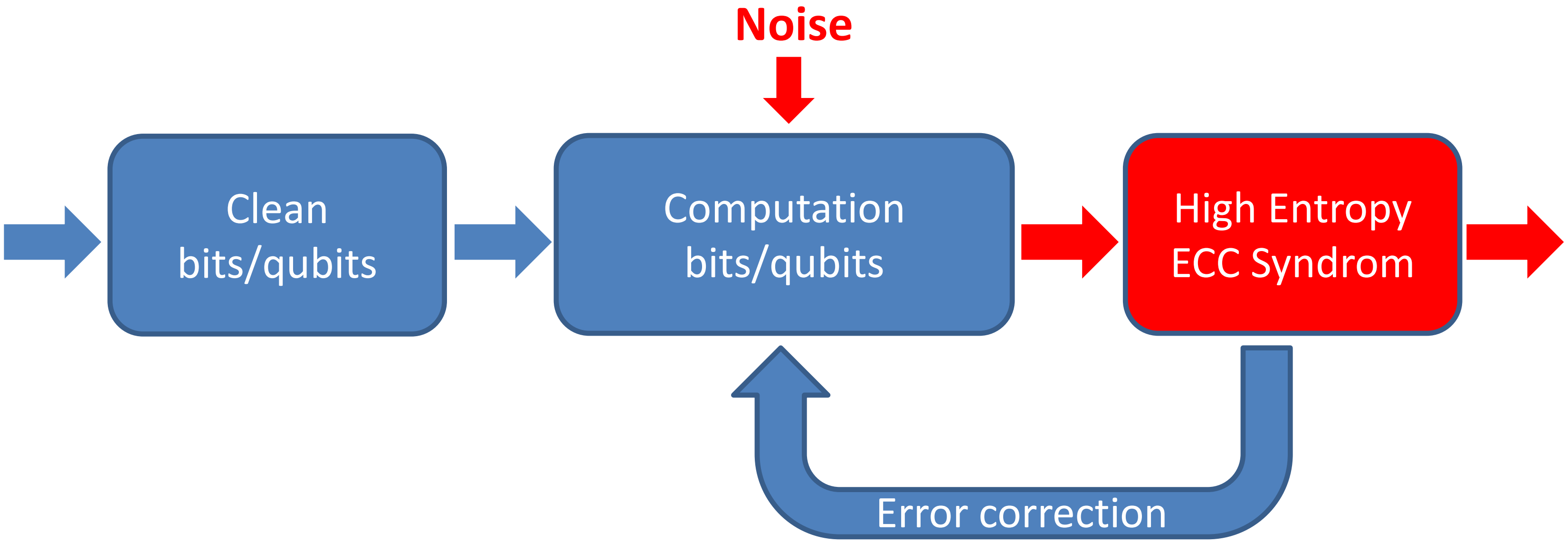

General fault-tolerant computation requires continuous supply of fresh clean bits/qubits

Fault-Tolerant/Noisy Computation Without Fresh Bits/Qubits?

Depolarizing Noise [ABIN96]: $\rho \rightarrow (1-p)\cdot\rho + p\cdot\frac{I}{2}$

- Can compute for $T = \tilde{\theta}(\log n)$ steps:
  Run standard error-correcting computation on part of the system
  Run simple purification on remaining unused qubits to obtain "clean qubits"
  Continue until all remaining qubits are used.

- Cannot compute for more than $T = O(\log n)$ steps:

$$I(X) = n - S(X)$$

$$I(X_0) = n$$

$$\text{Show}: \quad I(X_{t+1}) \le (1-p)\cdot I(X_t)$$

$$\text{For } T = O(\log n) \quad I(X_T) \le \varepsilon, \text{ so } S(X_T) \approx n$$

$$\text{and the full state is completely random and useless.}$$

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Classification of Quantum 1-Qubit Channels [KR01]

Bloch Sphere representation $\rho = \frac{1}{2}(I + w \cdot \sigma)$

where σ is the vector of Pauli matrices (X, Y, Z)

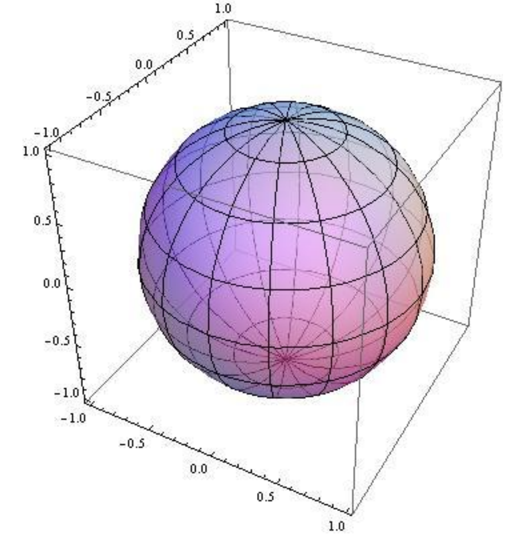
and w is a real vector of norm ≤ 1.

Up to unitaries $C(\frac{1}{2}[I + w \cdot \sigma]) = \frac{1}{2}[I + (t + T \cdot w) \cdot \sigma]$

T diagonal $\mathrm{T} = \begin{pmatrix} \lambda_x & & \\ & \lambda_y & \\ & & \lambda_z \end{pmatrix}$

where $|\lambda_y \pm \lambda_z| \leq |1 \pm \lambda_x|$ etc.

Unital Channels have $t = 0$, and $C(\frac{1}{2} I) = \frac{1}{2} I$



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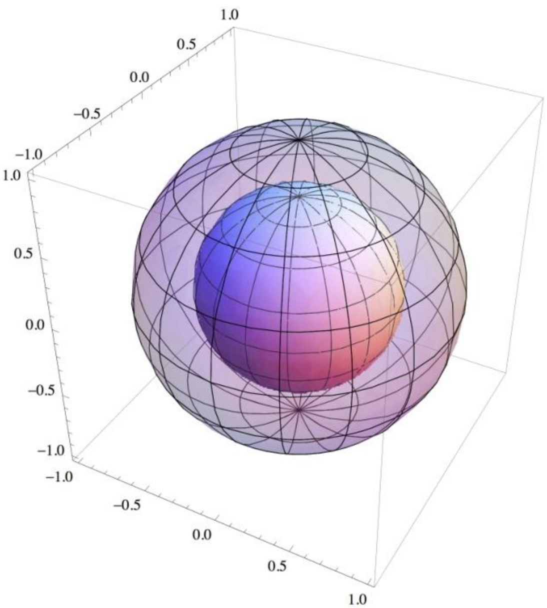
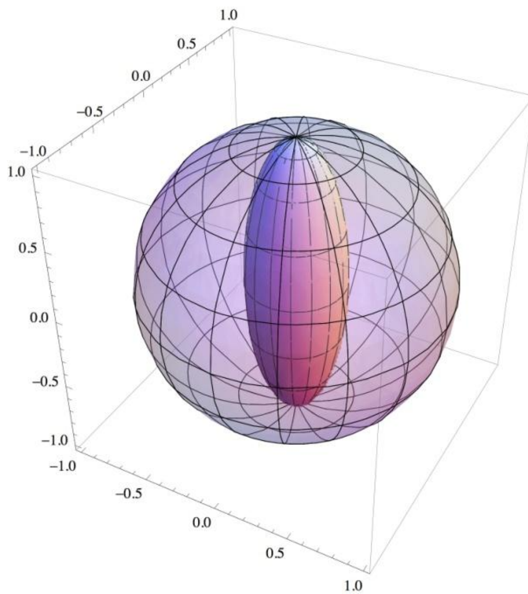


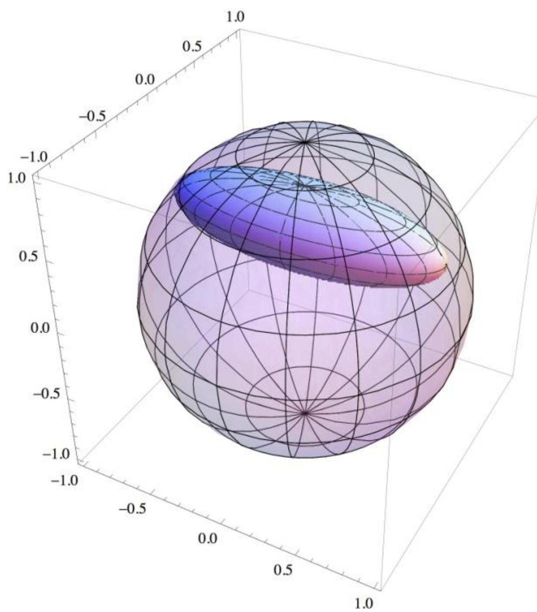
Depolarizing
Shrinking
to zero

$$\rho \to (1-p)\cdot\rho + p\cdot\tfrac{I}{2}$$



Dephasing
Shrinking
to a diameter

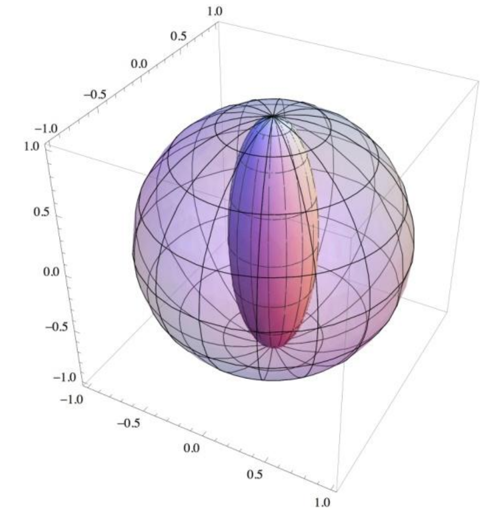
$$\rho \to (1-p)\cdot\rho + p\cdot Z\rho Z$$



Non Unital
Shrinking
to a point ≠ 0

$t \neq 0$

Dephasing Channel Noise for *n*-Qubit Register



- **Polynomial Computation:**

  Can compute on n^{α} qubits for n^{β} steps if $\alpha \cdot \beta < 1$.

  Proof: Each step use a block of n^{α} clean qubits.

- No bounds on classical computation.
- Can keep an EPR pair up to $O(n/\log n)$ time steps.

- **Polynomial Upper Bound:**

  Cannot keep entanglement more than $O(n^3)$ time.

  Proof: Channel does not decrease entropy.

  If entropy does not grow state is close to invariant state.
  Invariant states of register are diagonal so state has almost no entanglement.

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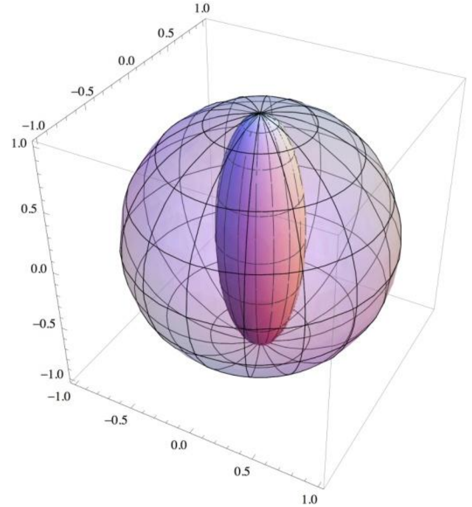

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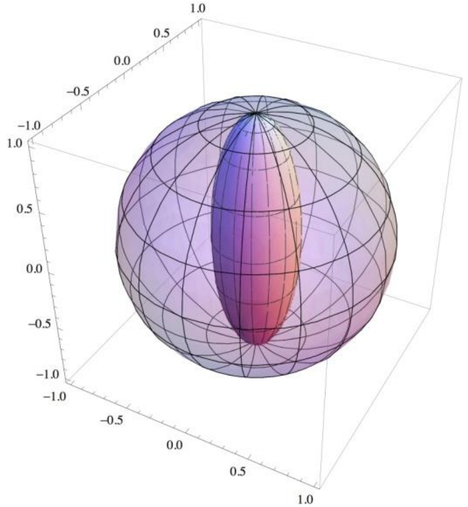

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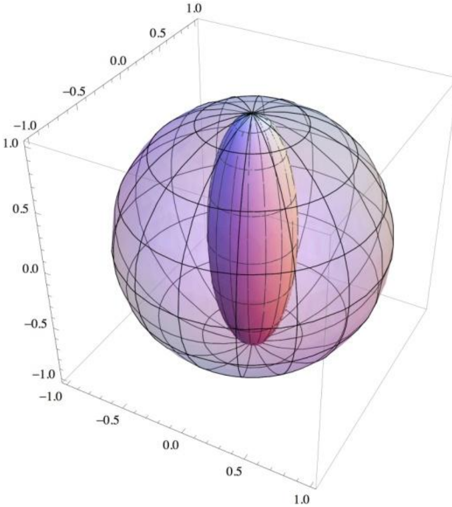

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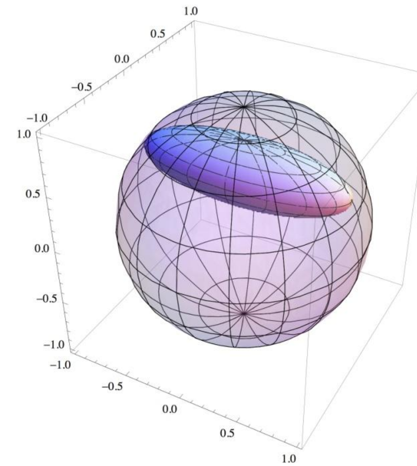
Non Unital Channel Noise

The Quantum Refrigerator: If noise below thershold we can run a computation of depth/time D on n qubits

- using $O(n \text{ polylog}(nD))$ qubits
- and $O(D \text{ polylog}(nD))$ computation steps.



- If D is exponential in n then the fault-tolerant system has polynomial overhead.

- Amplitude Damping channel with poly(n) qubits cannot compute more than exponential time.

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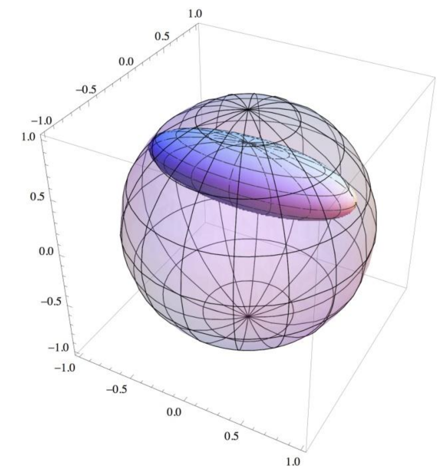

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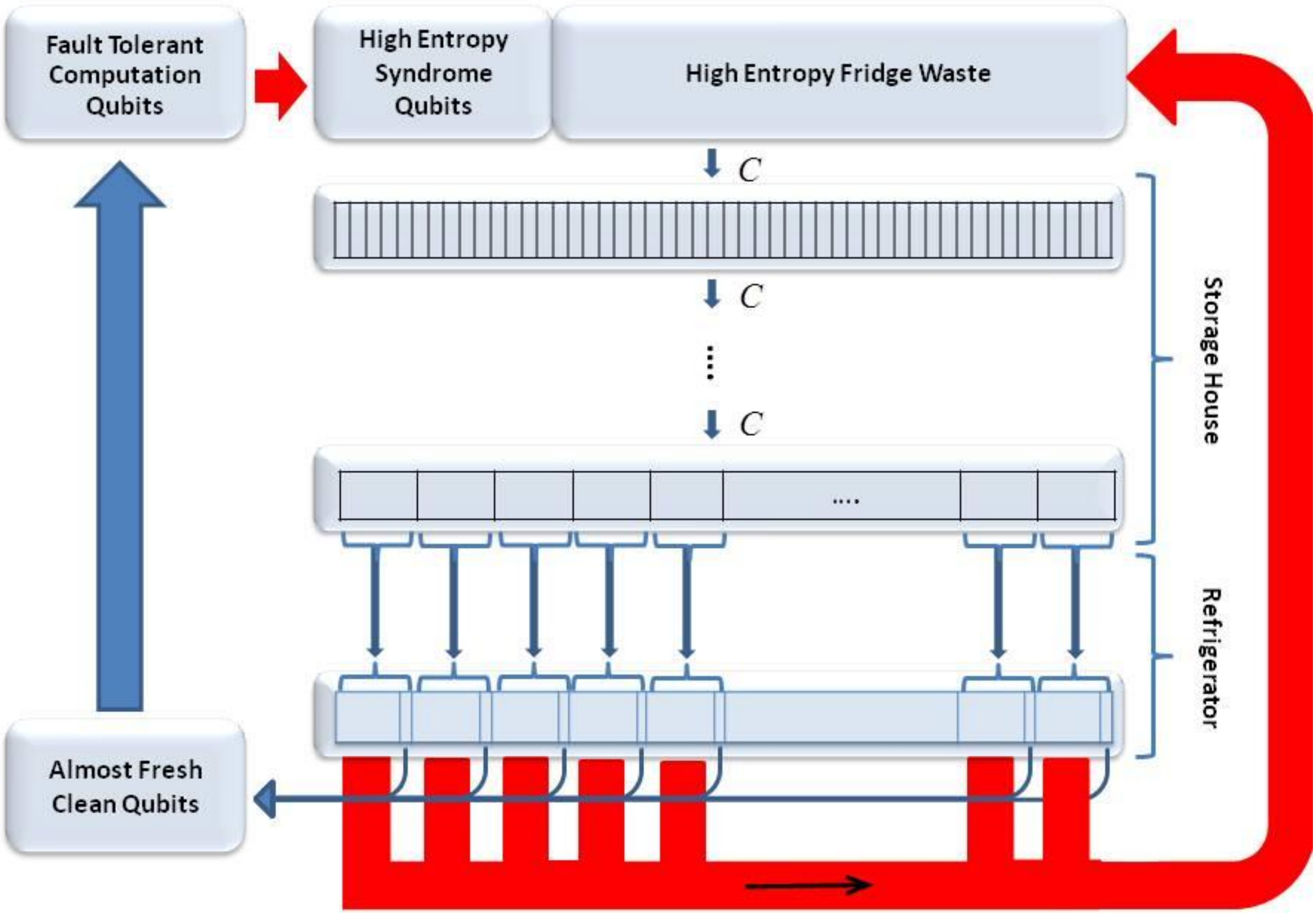
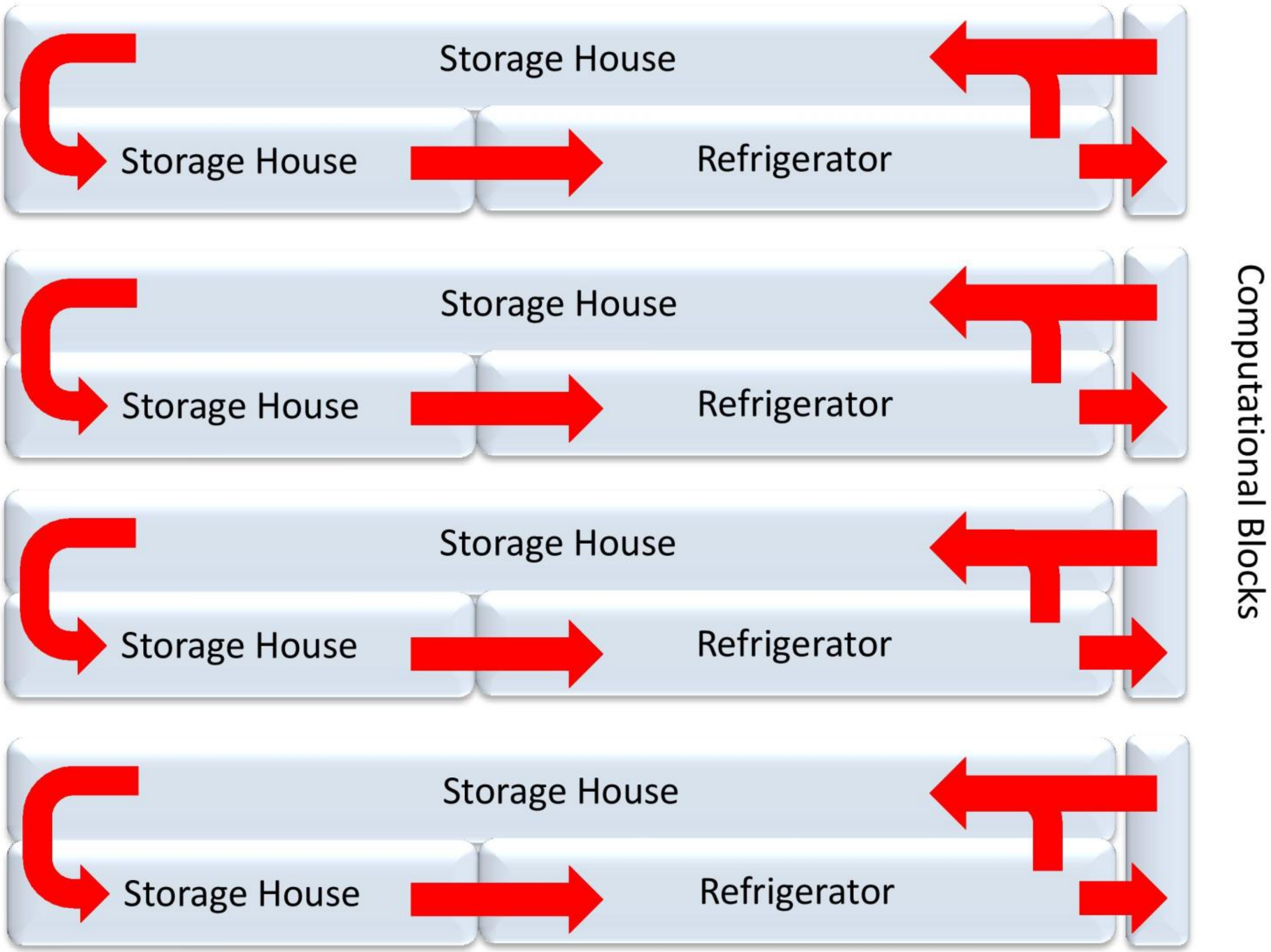


The Quantum Refrigerator in 2-Dimensions



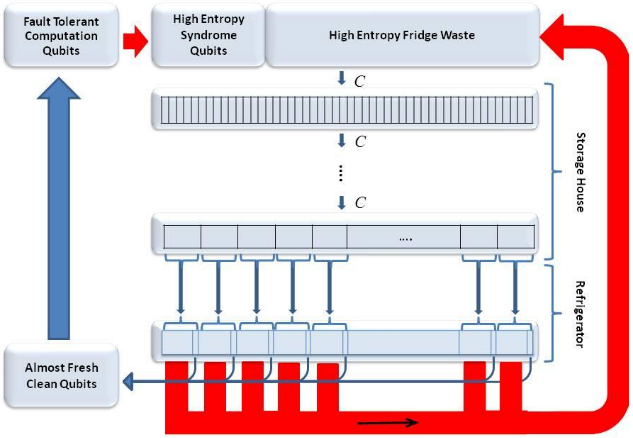
Quantum Refrigerator - Summary

We identify 3 types of behavior for independent 1-qubit noise, without supply of clean qubits (and polynomial overhead):

- **Depolarizing type:** Only logarithmic depth computation is possible.
- **Depashing type:** Polynomial length computation
- **Non Unital type:** Exponential length computation is possible.

Open Problems:

- Tighten upper bounds for dephasing channel
- Exponential upper bound for all non unital channels.
- Qudits of dimension > 2.



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