Finite blocklength converse bounds for quantum channels (arXiv:1210.4722)

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Introduction: Codes

Classical data over quantum channels.

Entanglement-assisted (EA) code $\mathcal{Z}$ of **size** M:

For uniform source $\mathcal{S}_M$ $(\Pr(W = w|\mathcal{S}_M) = 1/M)$:
Average input $\rho_\mathrm{A} = \frac{1}{M}\sum_{w=1}^{M}\rho(w)_\mathrm{A}$.
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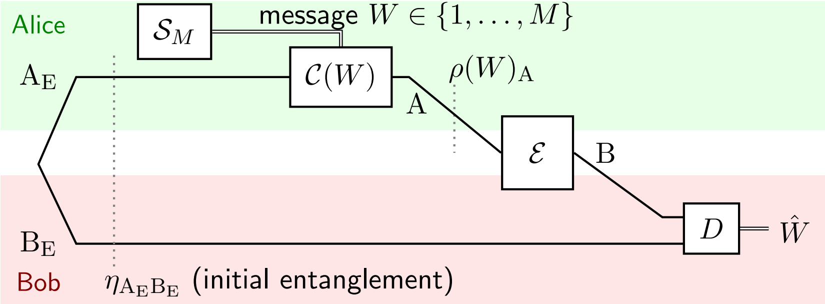

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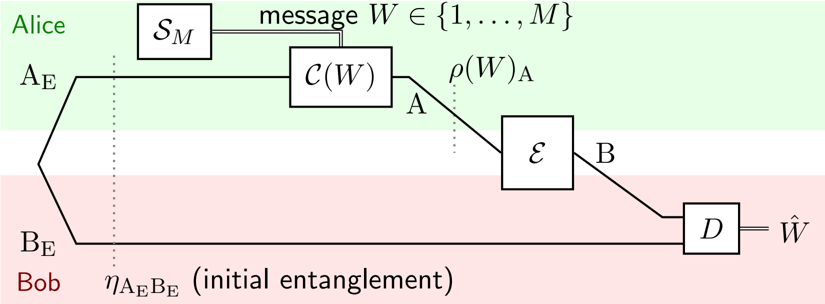

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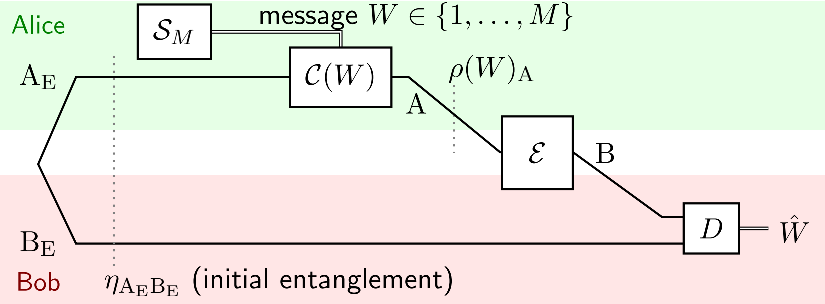

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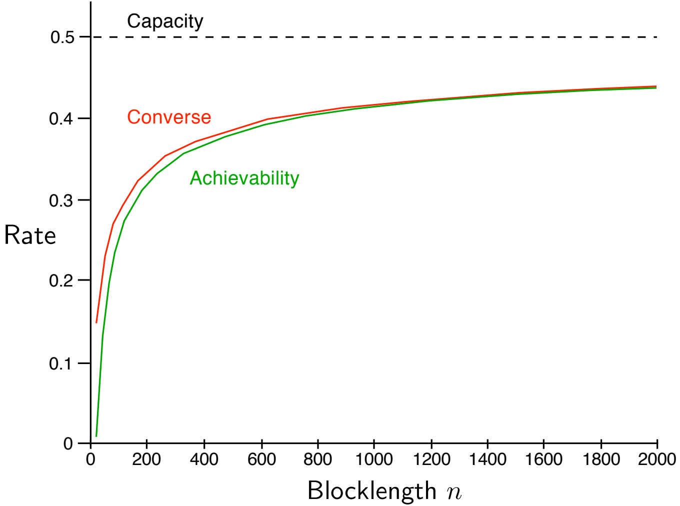
Background and motivation

Converse and achievability bounds[1] on the rate $\frac{1}{n} \log M_\epsilon(\mathcal{E}^n)$ when $\epsilon = 1/1000$ and $\boldsymbol{\mathcal{E}}$ is the BSC with $\Pr(\text{bit flip}) = 0.11$.



[1] Polyanskiy, Poor, Verdú. IEEE Trans. Inf. T., 56, 2307-2359

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- Datta & Hsieh (arXiv:1105.3321) give converse (and achievability) for $M_\epsilon^{\mathrm{E}}(\mathcal{E})$, but it has some disadvantages (diverges as $\epsilon \to 0$; not clear how to compute).
- Polyanskiy–Poor–Verdú gives a classical converse which relates coding to hypothesis testing. It is simple, and powerful enough to derive many important classical converse bounds, so we want a quantum generalisation.
- The converse in Wang & Renner (arXiv:1007.5456) for $M_\epsilon(\mathcal{E})$ is almost such a generalisation for unassisted codes (see also Hayashi's book).
- We obtain a hierarchy of bounds based on quantum hypothesis testing of a bipartite system with restricted measurements, including a novel converse for EA codes, and a generalisation of Wang-Renner converse for unassisted codes.

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Test T for H_0: The element of a binary POVM $\{T, \mathbb{1} - T\}$ for the outcome "accept H_0".

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- $\mathbf{L}$: *Local tests* - Test on joint outcome of local measurements (coordinated only by shared randomness).
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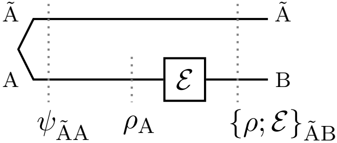

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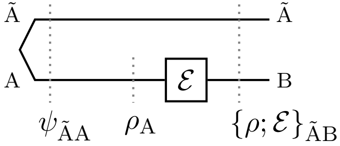

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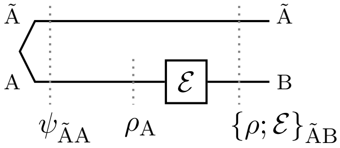

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Suppose there exists a code of size M, average input ρ, with error ϵ for $\mathcal{E}$, which maps to a test $T_{\tilde{A}B}$ in class $\mathbf{\Omega}$.

H_0: State is $\{\rho;\mathcal{E}\}_{\tilde{A}B}$ H_1: State is $\rho^*_{\tilde{A}}\sigma_B$

• $\alpha(T_{\tilde{A}B}) = 1 - \mathrm{Tr}\{\rho;\mathcal{E}\}_{\tilde{A}B}T_{\tilde{A}B} = \epsilon$.
For a CPTP map $\mathcal{F}$ with constant output σ_B the success probability is $1/M$, so
• $\beta(T_{\tilde{A}B}) = \mathrm{Tr}\rho^*_{\tilde{A}}\sigma_B T_{\tilde{A}B} = 1/M$.

Therefore $\forall\sigma_B : \beta^{\mathbf{\Omega}}_\epsilon(\{\rho;\mathcal{E}\}_{\tilde{A}B}, \rho^*_{\tilde{A}}\sigma_B) \le 1/M$, or

$$M \le B^{\mathbf{\Omega}}_\epsilon(\rho,\mathcal{E}) := \left(\max_\sigma \beta^{\mathbf{\Omega}}_\epsilon(\{\rho;\mathcal{E}\}_{\tilde{A}B}, \rho^*_{\tilde{A}}\sigma_B)\right)^{-1}$$

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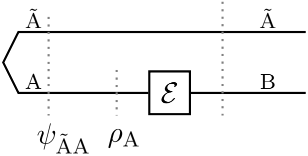

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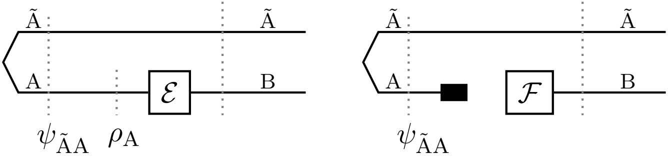

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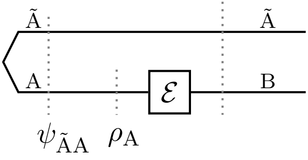

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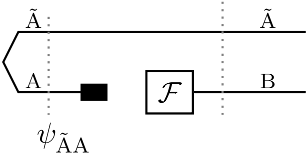

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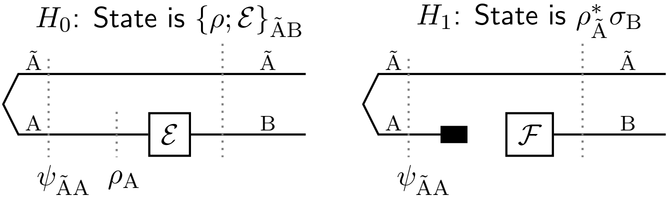

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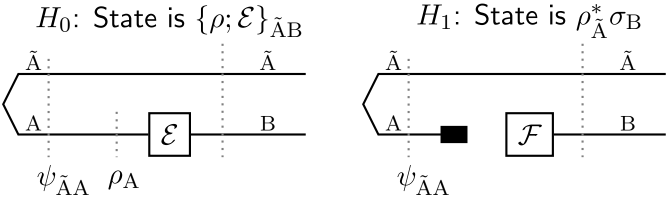

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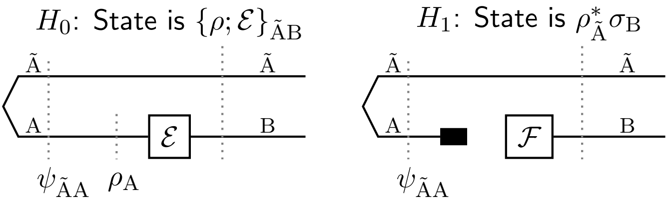

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For unassisted codes: Since these map to local tests, for any class $\mathbf{\Omega}$ containing $\mathbf{L}$, $M_\epsilon(\rho, \mathcal{E}) \leq B_\epsilon^{\mathbf{\Omega}}(\rho, \mathcal{E})$, and

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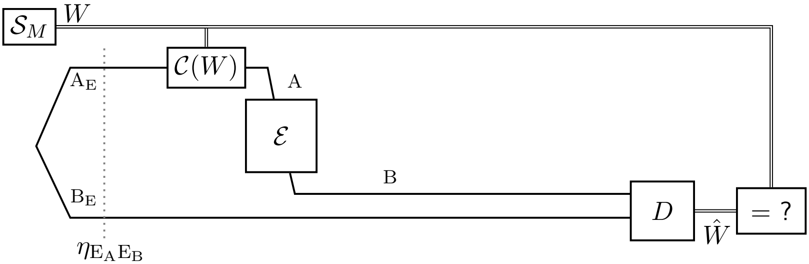

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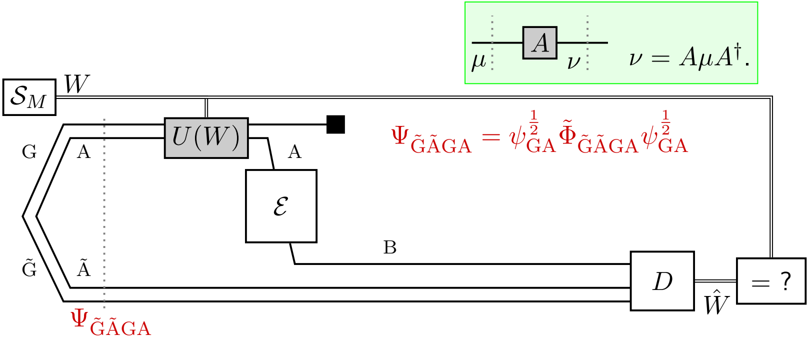

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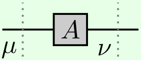
$$\rho_{\mathrm{A}} = \frac{1}{M}\sum_w \mathrm{Tr}_{\mathrm{G}} U(w)_{\mathrm{GA}} \psi_{\mathrm{GA}} U(w)^{\dagger}_{\mathrm{GA}}$$



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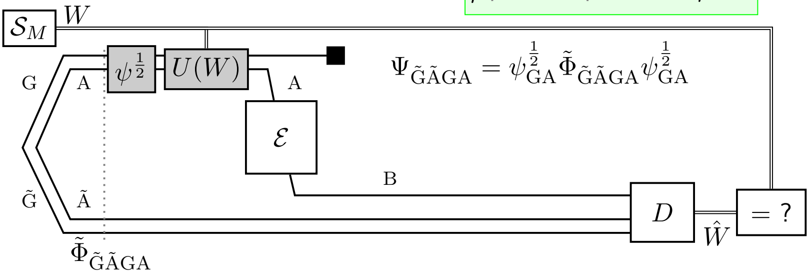

$$\Psi_{\tilde{\mathrm{G}}\tilde{\mathrm{A}}\mathrm{GA}} = \psi^{\frac{1}{2}}_{\mathrm{GA}} \tilde{\Phi}_{\tilde{\mathrm{G}}\tilde{\mathrm{A}}\mathrm{GA}} \psi^{\frac{1}{2}}_{\mathrm{GA}}$$

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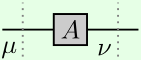

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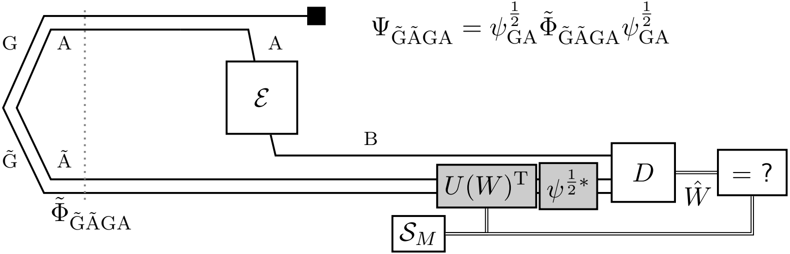

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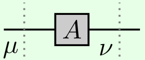

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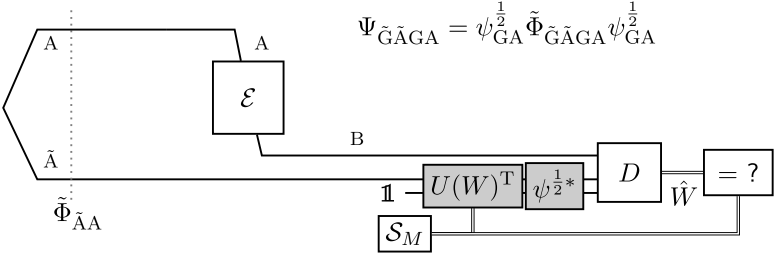

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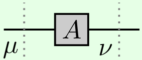

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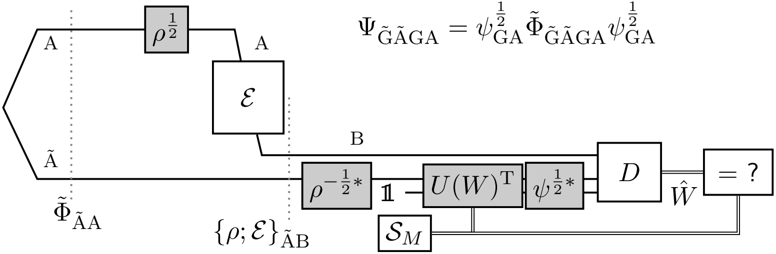

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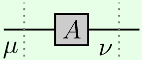

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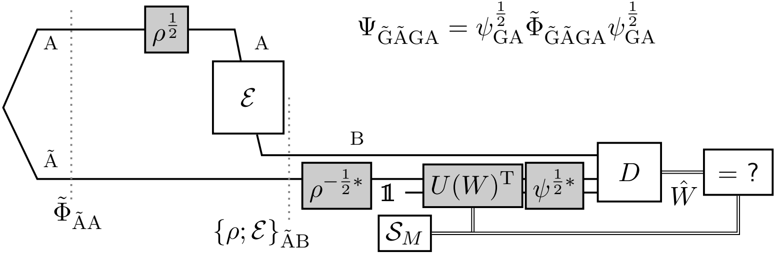

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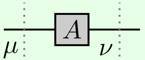

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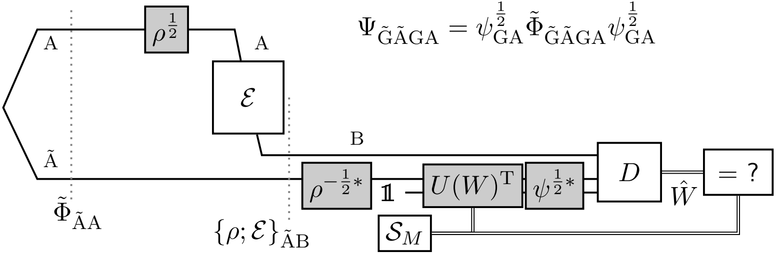

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Properties of the bounds

- They reduce to the PPV converse for classical channels.
- Entanglement-assisted bound:
  - Can recover converse part of BSST.
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Example: EA coding over the depolarising channel

- d-dimensional depolarising channel:
  $\mathcal{D}[\tau] = (1-p)\tau + p\mathrm{Tr}(\tau)\mu$,
  where $\mu = \mathbb{1}/d$ is the maximally mixed state.
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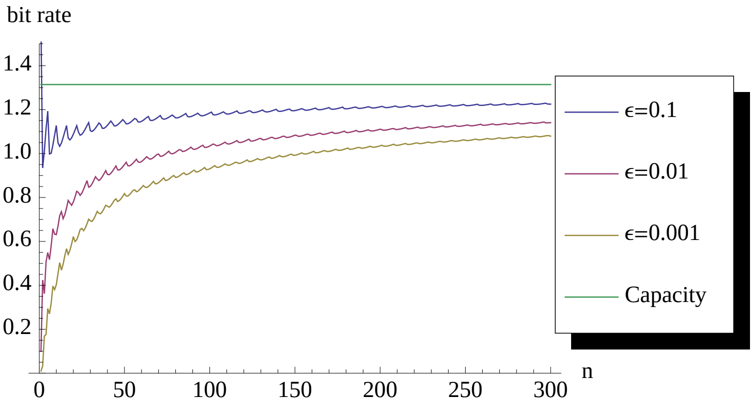

Figure: The upper bound on the rate for entanglement assisted codes over the p=0.15 depolarising channel for three different error probabilities.

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