Superactivation of quantum nonlocality

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What is quantum nonlocality?

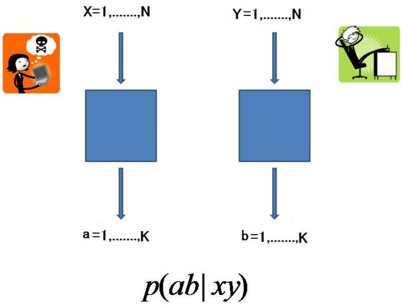

Figure: Alice and Bob measurements. Inputs: x and y, Outputs: a and b. $P(a, b|x, y)$ is the probability of obtaining the pair (a, b) when Alice and Bob measure, respectively, with the input x and y.

We deal with the "probability distributions"

$$P = (P(a, b|x, y))_{x,y=1,\cdots,N}^{a,b=1,\cdots,K}.$$

What is quantum nonlocality?

- $P = \{P(a, b|x, y)\}_{x,y;a,b}$ is a Classical prob. distribution ($P \in \mathcal{L}$) if it is in the convex hull of the elements of the form

$$P(a, b|x, y) = P_1(a|x)P_2(b|y) \text{ for every } x, y, a, b, \text{ where}$$

$$P_1(a|x) \geq 0 \text{ and } \sum_a P_1(a|x) = 1 \text{ for every } x \text{ (similar for } P_2\text{)}.$$

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- P is a Quantum prob. distribution ($P \in \mathcal{Q}$) if it can be written like:

$$P(a,b|x,y) = tr(E_x^a \otimes F_y^b \rho) \text{ for every } x,y,a,b, \text{ where}$$

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- Quantum nonlocality: $\mathcal{L} \subsetneq \mathcal{Q}$.

Why is quantum nonlocality interesting?

- Fundamental and fascinating phenomenon!

IDEA: THEORY $\leftrightsquigarrow$ EXPERIMENTAL DATA

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- It has many applications:
  1. Device Independent Quantum Cryptography.
  2. Generation of random numbers.
  3. Interactive proof systems.

  ... ...

How to study nonlocality? Bell inequalities

- Given $M = \{M_{x,y}^{a,b}\}_{x,y;a,b}$ and $P = \{P(a,b|x,y)\}_{x,y;a,b}$, denote

$$\langle M, P\rangle = \sum_{x,y,a,b} M_{x,y}^{a,b} P(a,b|x,y) \quad \text{and} \quad \omega(M) = \sup_{P\in\mathcal{L}} |\langle M, P\rangle|.$$

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Then,

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- $LV(\rho) \geq 1$ for ever ρ. Moreover, ρ local $\Leftrightarrow$ $LV(\rho) = 1$.

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- Behavior of $LV(\rho)$... asymptotically .... at least for some states...

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- Superactivation of nonlocality:

  Find a local state ρ such that $LV(\rho^{\otimes k}) > 1$ for some k.

An upper bound for $LV(\rho)$

- Theorem: Given any pure state $|\psi\rangle = \sum_{i=1}^{n} \alpha_i |ii\rangle$, $\alpha_i \geq$ we have

$$LV(|\psi\rangle) \leq UB(|\psi\rangle) := \Big(\sum_{i=1}^{n} \alpha_i\Big)^2.$$

In particular, $LV(\rho) \leq n$ for every n dimensional quantum state ρ.

Note that $UB(|\varphi_n\rangle) = n$.

Lower bound and the Khot and Visnoi game

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- There exit a Bell inequality $M(n) = \{M(n)_{x,y}^{a,b}\}_{x,y,a,b}$ and some POVMs $\{E(n)_x^a\}_{x,a}$, $\{F(n)_y^b\}_{y,b}$ such that
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- In fact,

$$C\frac{UB(|\psi\rangle)}{(\log n)^2} \leq LV(|\psi\rangle) \leq UB(|\psi\rangle)$$

for every n dimensional pure state $|\psi\rangle$ (even better!).

A tighter upper bound

- Theorem: For every n,

$$LV(|\varphi_n\rangle) \leq D\frac{n}{\sqrt{\log n}}.$$

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- We understand better the quantity $LV(|\varphi_n\rangle)$.
- Interesting part of the result: Proof

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- On the other hand, applying the KV game we have

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- Therefore,

$$\frac{LV(|\varphi_n\rangle^{\otimes_5})}{LV(|\varphi_n\rangle)^5} \geq \frac{Cn^5(\log n)^{\frac{5}{2}}}{D^5 n^5 (5\log n)^2} = C'\sqrt{\log n}.$$

Extending the argument

- Let us consider

$$\eta = \lambda|\varphi_n\rangle\langle\varphi_n| + (1-\lambda)\frac{1}{n^2}.$$

Note that

$$\eta^{\otimes_k} = \lambda^k|\varphi_{n^k}\rangle\langle\varphi_{n^k}| + \cdots\cdots$$

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- Since G_{n^k} has positive coefficients we know that

$$LV(\eta^{\otimes_k}) \geq \frac{1}{\omega(G_{n^k})}\langle G_{n^k}, P\rangle \geq C\lambda^k \frac{n^k}{(\log n^k)^2} = C\frac{(\lambda n)^k}{k^2(\log n)^2}.$$

Consequences

- Superactivation of quantum nonlocality:

  J. Barrett showed that there exist local isotropic states $\eta = \lambda|\varphi_n\rangle\langle\varphi_n| + (1-\lambda)\frac{1}{n^2}$ for $\lambda > \frac{1}{n}$. Therefore,

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- Unbounded almost-superactivation of quantum nonlocality:

  For every $\epsilon >$ and $\delta > 0$ there exists a high enough n such that $\eta = \lambda_n|\varphi_n\rangle\langle\varphi_n| + (1-\lambda_n)\frac{1}{n^2}$ verifies that

$$LV(\eta) \leq 1 + \epsilon \quad \text{and} \quad LV(\eta^{\otimes_5}) \geq \delta.$$

Thank you very much!