Learning-graph-based quantum algorithm for k-distinctness

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Introduction

Bounded 1-certificate complexity

We saw (the first talk) that for a function without any structure, e.g.,

k-sum problem:

Given $x_1, \ldots, x_n \in [q]$, detect whether there exist pairwise distinct $a_1, \ldots, a_k$ such that $x_{a_1} + x_{a_2} + \cdots + x_{a_k}$ is divisible by q.

quantum walk on the Johnson graph gives $O(n^{k/(k+1)})$ queries, and this is optimal.

Structure

In Miklos Santha's talk, we saw that if there is additional structure (not all certificate positions are allowed), we can do better, e.g.:

Triangle problem:

Given $x_{i,j} \in \{0,1\}$, with $1 \leq i < j \leq n$, detect whether there exist $1 \leq a < b < c \leq n$ such that $x_{a,b} = x_{a,c} = x_{b,c} = 1$.

Can be done with learning graphs in $O(n^{9/7})$ quantum queries.
Better than $O(n^{3/2})$ that would be possible without the structure.

Main Question

Simplification II: Only consider the *positions* of certificates inside the input string.

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What if we consider the values of the variables as well?

Plus: We can pursue consistent certificates, and drop inconsistent ones, thus, reducing the complexity.

Minus: Greater diversity makes the algorithm harder to analyze.

Values

Considering values, we certainly can do better:

k-threshold problem:

Given $x_1, \ldots, x_n \in \{0,1\}$, detect whether $\sum_{i=1}^{n} x_i \geq k$.

- Can be easily solved in $O(\sqrt{n})$ queries using Grover search.

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- Can be easily solved in $O(\sqrt{n})$ queries using Grover search.
- Well... it's too simple.

k-distinctness problem

We arrive at our main problem:

k-distinctness problem:

Given $x_1, \ldots, x_n \in [q]$, detect whether there exist $a_1, \ldots, a_k$, all distinct, such that $x_{a_1} = x_{a_2} = \cdots = x_{a_k}$.

- Quantum walk algorithm solving the problem in $O(n^{k/(k+1)})$ queries.
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- We developed a quantum algorithm with query complexity

$$O\left(n^{1-2^{k-2}/(2^k-1)}\right) = o(n^{3/4}).$$

Pursuing consistent certificates

How does it look like?

Similarly as in Miklos Santha's talk for Element Distinctness.

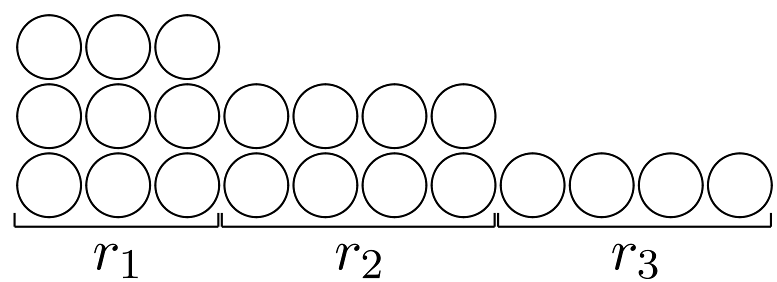
Let $a_1, \ldots, a_k$ be a 1-certificate in the input.

The last k steps in the learning graph are as on the right:

$\vdots$
Load a_1
Load a_2
$\vdots$
Load a_k

Assume before that the vertices of the learning graphs ($\subseteq [n]$) contain

r_1 unique elements, r_2 pairs of equal elements, …, r_{k-1} $(k-1)$-tuples of equal elements.



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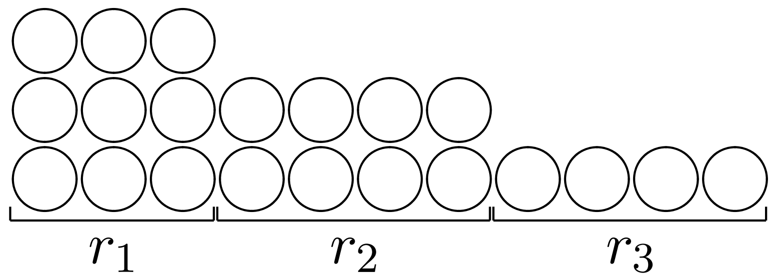

The complexity of loading $a_1, \ldots, a_k$ is $O(n/\sqrt{\min\{r_1, \ldots, r_{k-1}\}})$.

Proof. As for element distinctness: When a_i is loaded, $(i-1)$-tuple of equal elements $\{a_1, \ldots, a_{i-1}\}$ is hidden among $r_{i-1}+1$ such tuples in a vertex of the learning graph. □

Preparation of the state

In the quantum walk on the Johnson graph algorithm, $S \subseteq [n]$ is chosen uniformly at random from subsets of size r.
Thus, r_{k-1} is very small: $O(n \cdot r^{k-1}/n^{k-1})$.

Using the values, we can "distill" subsets containing large number of large tuples of equal elements.

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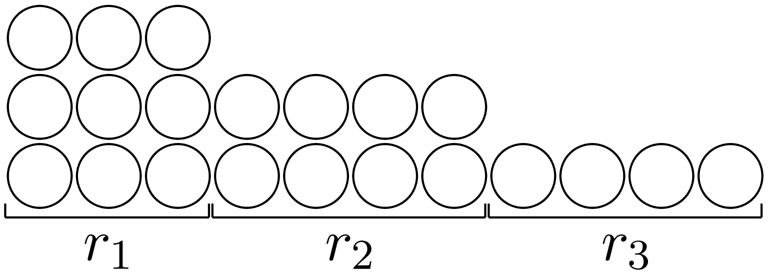
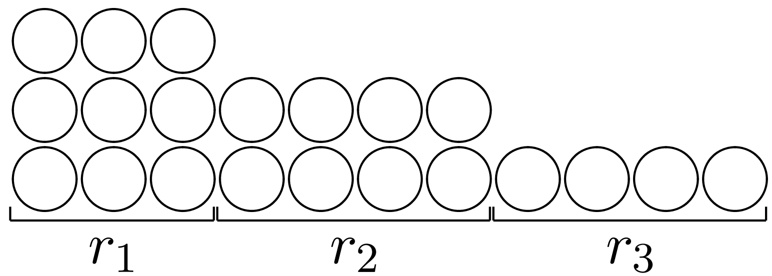
Related Question
What is the complexity of preparing the uniform superposition over all $S \subseteq [n]$ of the form



Preparation of the state

Preparation of uniform superposition over all $S \subseteq [n]$ that contain



Tentative Plan

1.	Start with the uniform superposition of $(r_1 + \cdots + r_{k-1})$-subsets.
2.	Find $r_2 + \cdots + r_{k-1}$ elements equal to elements in the current set.
3.	Find $r_3 + \cdots + r_{k-1}$ elements equal to two elements in the current set.
	$\vdots$
$k-1$.	Find r_{k-1} elements equal to $k-2$ elements in the current set.

Preparation of the state

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We may assume there is unique k-tuple of equal elements in any positive input.
We may assume there are $\Omega(n)$ $(k-1)$-tuples of equal elements.
Assume also $r_1 > r_2 > \cdots > r_{k-1}$.

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Assume also $r_1 > r_2 > \cdots > r_{k-1}$.
Then, complexity of preparing the state is:

$$r_1 + r_2\sqrt{\frac{n}{r_1}} + r_3\sqrt{\frac{n}{r_2}} + \cdots + r_{k-1}\sqrt{\frac{n}{r_{k-2}}}.$$

Total complexity

Assume $r_1 > r_2 > \cdots > r_{k-1}$.
Complexity of preparing the uniform superposition is:

$$r_1 + r_2\sqrt{\frac{n}{r_1}} + r_3\sqrt{\frac{n}{r_2}} + \cdots + r_{k-1}\sqrt{\frac{n}{r_{k-2}}}.$$

Complexity of the final stage

$$n/\sqrt{r_{k-1}}.$$

Total complexity is optimized to

$$O\left(n^{1-2^{k-2}/(2^k-1)}\right) = o(n^{3/4}).$$

Diversity

Preparation of the state

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This algorithm does not generate the uniform superposition, nor a state close to it!

Preparation of the state

Assume both states have amplitudes α.

Perform Grover search for an element making a pair with an element in S.

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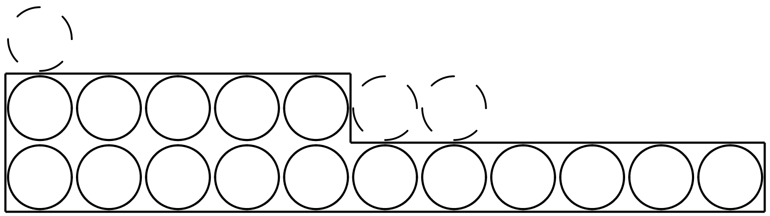

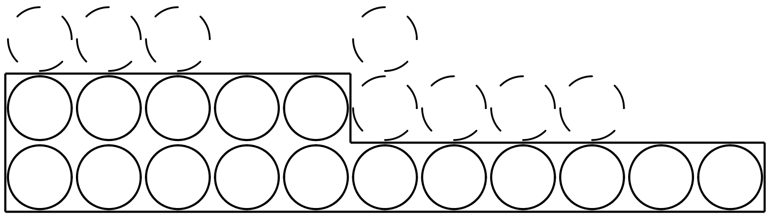

Assume both states have amplitudes α.

Perform Grover search for an element making a pair with an element in S.

Assume the Grover search works perfectly for both subsets.
Then the amplitude is subdivided into:

$\alpha/\sqrt{2}$ $\qquad$ $\alpha/\sqrt{5}$

This accumulates with each step, and we get an exponential bias.

Summary

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- We saw gains and losses of using values of the variables.
- These problems can be solved for k-distinctness, but I will not go into the detail.

Open Problem

- Obtain a similar framework for these types of problems, as it was done in the first presentation (learning graphs).
- Prove matching lower bound for k-distinctness.

Thank you!