

An improved 1D area law for frustration-free systems

Itai Arad*

Zeph Landau[†]

Umesh Vazirani[‡]

August 31, 2011

Extended Abstract

Hastings' 1D area law [1] is arguably one of the most important results in quantum Hamiltonian complexity in recent years. It states that the entanglement entropy in the ground state of gapped 1D local system is bounded by a constant, independent of the system size. Specifically, for a nearest neighbor system on a chain $H = \sum_{i=1}^{n-1} H_i$ with particle dimension d , interaction strength $\|H_i\| \leq J$ and a spectral gap $\epsilon > 0$, the theorem states that entanglement entropy across any cut in the chain is upper bounded by $S \leq e^{\mathcal{O}(X)}$ for $X \stackrel{\text{def}}{=} \frac{J \log d}{\epsilon}$.

Hastings' result implies that ground states of gapped 1D systems are in some sense classical. Their limited amount of entanglement implies that they can be well-approximated by states that admit a classical description of a polynomial size (MPS), which can in turn be used to approximate any local observable efficiently on a classical computer. From a practical point of view, this gives us a solid theoretical understanding of why variational methods such as DMRG work so well on these systems. From a complexity-theoretic point of view it tells us that the local Hamiltonian problem for such 1D systems with a constant spectral gap is inside NP. In fact, it means that the problem is inside NP as long as that gap is above $\mathcal{O}((\log \log n)^{-1})$, where n is the size of the system, or alternatively that DMRG algorithms are expected to work well for these type of systems.

Two major issues remained open following Hastings' paper. The first is an extension of his results to 2 and 3 dimensional systems. The second was the dependence of the bound on X . Hastings shows that S scales exponentially in X , whereas the best lower bounds come from specially crafted 1D Hamiltonians [2, 3], scale like $\epsilon^{-1/4}$. This is important for two reasons: the dependence on ϵ determines how close to the critical point ($\epsilon = 0$) the ground state admits a polynomial size MPS, and by implication the viability of variational methods for such systems. At the same time, improving the dependence on $\log d$ provides a possible path towards proving an area law in higher dimensions. Indeed, a bound on S that scales as $\mathcal{O}(\log d)$ would imply higher dimensional area laws by the naive reduction from a D -dimensional system to a 1-D system by fusing together particles on surfaces parallel to the boundary.

A new approach to proving the area law for 1D frustration free systems (i.e., systems where the ground state also minimizes the energy of every local term), was introduced in Ref. [4]. The proof

*School of Computer Science and Engineering, The Hebrew University, Jerusalem, Israel

[†]Department of Computer Science, UC Berkeley

[‡]Department of Computer Science, UC Berkeley

replaced Hastings' analytical machinery, including the Lieb-Robinson bound and spectral Fourier analysis, with the Detectability Lemma [5], a combinatorial lemma about local Hamiltonians. This resulted in a somewhat cleaner and simpler proof, but nevertheless, the overall structure of the two proofs was identical, and consequently the entropy bound was the same, i.e., $S \leq e^{\mathcal{O}(X)}$.

In this paper we give a new proof of the 1D area-law for frustration-free systems. Following Ref. [4], the new proof uses the detectability lemma as its starting point. Unlike Ref. [4], however, it uses a very different approach from that of Hastings' proof, and consequently it yields an exponentially better bound on the entanglement entropy. We prove:

Theorem 1 *Let $|\Omega\rangle$ be the ground state of a frustration-free, nearest neighbor Hamiltonian system $H = \sum_{i=1}^n H_i$ on a 1D chain of n particles of dimension d . Assume that the system has spectral gap $\epsilon > 0$, and an interaction strength $\|H_i\| \leq J$. Then along any cut in the chain, the entanglement entropy of $|\Omega\rangle$ is bounded by*

$$S(\Omega) \leq \mathcal{O}(1) \cdot X^3 \log^8 X, \quad (1)$$

for $X \stackrel{\text{def}}{=} \frac{J \log d}{\epsilon}$.

Our result narrows to a polynomial factor the gap between the upper and lower bounds on the entanglement entropy as a function of the spectral gap ϵ . In addition, in 2 or more dimensions, we were able to utilize some of the local structure of the problem *along the boundary surface*, and prove that

Theorem 2 *In higher dimensions, the entanglement entropy in the ground state between a contiguous region L and the rest of the system is bounded by*

$$S_L(\Omega) \leq \mathcal{O}(1) \cdot |\partial L|^2 \log^2 X^3 \log^8(|\partial L| \cdot X). \quad (2)$$

This bound is at the cusp of being non-trivial; any further improvement that would bound the entropy by $|\partial L|^{2-\delta}$ for any $\delta > 0$, would prove a sub-volume law for 2D.

Our results are proved only for the frustration-free case. Nevertheless, we believe that they may be generalizable to the frustrated case. Indeed, Hastings' original proof [1] essentially reduces the frustrated case to an approximately frustration-free system by coarse graining, and a similar approach may also work here. Additionally, proving the area law in 2 or more dimensions remains an extremely important open problem even in the frustration-free case.

Outline of the proof

The key to bounding the entropy across a cut is finding a product state $|\phi_0\rangle = |\phi_L\rangle \otimes |\phi_{\bar{L}}\rangle$ with respect to the bi-partitioning of the system, which has large overlap with $|\Omega\rangle$. Our approach to finding such a product state is to start with any product state with non-zero overlap with $|\Omega\rangle$, and act on it with an operator that increases its overlap with $|\Omega\rangle$, without increasing its Schmidt rank (SR) much. i.e., we construct an operator K with the following property: K fixes $|\Omega\rangle$, but when applied to any state $|\psi\rangle$, it shrinks the component orthogonal to $|\Omega\rangle$ by a factor of Δ while increasing the SR of $|\psi\rangle$ by a factor of at most D . Clearly, there is a race between these two factors D and Δ . It turns out that when $D \cdot \Delta < 1/2$, we can amplify the overlap with $|\Omega\rangle$ by replacing $|\phi_0\rangle$ by one of the Schmidt vectors of

$K|\phi_0\rangle$. This amplification continues all the way until the overlap is $\sqrt{1/(2D)}$. A few more applications of K to this product state yield a state with Schmidt rank $D^{\mathcal{O}(1)}$, which has constant (D -independent) overlap with $|\Omega\rangle$. Further applications of K give rise to Schmidt coefficients with vanishing mass, and therefore the entanglement entropy of $|\Omega\rangle$ can be bounded by $\mathcal{O}(\log D)$.

Our starting point for constructing the operator K is the detectability lemma (DL). Let P_i denote the (local) projection into the ground space of the local Hamiltonian term H_i . We can partition the projections $\{P_i\}$ into two subsets of even and odd projections, which are called “layers”. Inside each layer, the projections commute because they are non-intersecting. Consequently, $\Pi_{\text{odd}} \stackrel{\text{def}}{=} P_1 \cdot P_3 \cdot P_5 \cdots$ and $\Pi_{\text{even}} \stackrel{\text{def}}{=} P_2 \cdot P_4 \cdot P_6 \cdots$ are the projections into the common eigenspace of the odd and even layers. Then according to the DL, the operator $A \stackrel{\text{def}}{=} \Pi_{\text{even}} \Pi_{\text{odd}}$ is an approximation to the ground state projection: it preserves the ground state, while shrinking its perpendicular space by an n -independent factor $\Delta_0(\epsilon) \simeq 1 - c\epsilon$ (where c is some geometrical factor). Moreover, each application of A increases the SR of our state by a constant factor of $D_0 \stackrel{\text{def}}{=} d^2$ (due to the projection that intersects with the cut in the chain). Unfortunately, we would expect $D_0 \Delta_0 \gg 1$, so the operator A does not by itself suffice to carry out our plan.

The proof then proceeds by modifying the operator A to decrease its SR factor while maintaining most of its shrinking factor. For concreteness, assume that the even layer contains the projection that intersects with the cut. We will focus on a segment of m projections around the cut, and denote their product by Π_m , so that $\Pi_{\text{even}} = \Pi_m \Pi_{\text{rest}}$. We will replace the operator Π_{even} with $\hat{\Pi}_m \Pi_{\text{rest}}$ that closely approximates Π_{even} while increasing the SR by much less than D_0 (when amortized over several applications).

To achieve this, consider what happens when we apply A^ℓ ; picture A^ℓ as a stack of 2ℓ layers that correspond to the alternating applications of Π_{even} and Π_{odd} . If we could somehow replace the product Π_m by a product that contains only rm projections for some $r \ll 1$ (not necessarily the same projections every time), then we would be able to find a column that contains at most $r\ell$ projections. The SR increase across that column would be at most $D_0^{r\ell}$, and as that column is at distance of at most m particles from the cut, it would imply that the SR at the cut increased by at most a factor of $D_0^{r\ell+m}$. The average SR factor is therefore $D_0^{r+m/\ell}$ which approaches D_0^r as ℓ increases. The central lemma of the paper is called the “diluting lemma”, and it does exactly this. We show how the problem of finding a diluted approximation for Π_m can be reduced to the classical problem of finding a low-degree polynomial $P(x)$ such that $P(0) = 1$ and for every $x \in [1, m]$, $|P(x)| < \delta < 1$ for some constant $0 < \delta < 1$. It turns out that the best polynomial to achieve this (i.e., the polynomial of the lowest degree), is the Chebyshev polynomial of degree \sqrt{m} , properly scaled to the region $[1, m]$. We use this polynomial to define $\hat{\Pi}_m$, the approximation of Π_m , which results in a diluting factor of $r = \frac{1}{\sqrt{m}}$. This translates into an SR factor of $D = D_0^{1/\sqrt{m}}$ at the price of slightly worsening the shrinking factor $\Delta_0(\epsilon)$. The end result is thus an operator K with factors (D, Δ) such that $D \cdot \Delta < 1/2$ as needed.

A clarification.

This work should be viewed as the union of the improved area law bound from Ref. [6], which was accepted to FOCS 2011, with subsequent ideas using Chebyshev polynomials to strengthen and simplify the results. We attach a copy of Ref. [6].

References

- [1] M. Hastings, “An Area Law for One Dimensional Quantum Systems,” *JSTAT, P*, vol. 8024, 2007.
- [2] S. Irani, “Ground state entanglement in one-dimensional translationally invariant quantum systems,” *Journal of Mathematical Physics*, vol. 51, no. 2, p. 022101, 2010.
- [3] D. Gottesman and M. B. Hastings, “Entanglement versus gap for one-dimensional spin systems,” *New Journal of Physics*, vol. 12, no. 2, p. 025002, 2010.
- [4] D. Aharonov, I. Arad, Z. Landau, and U. Vazirani, “Quantum Hamiltonian complexity and the detectability lemma,” *arXiv:1011.3445*, Nov. 2010.
- [5] D. Aharonov, I. Arad, Z. Landau, and U. Vazirani, “The detectability lemma and quantum gap amplification,” in *STOC '09: Proceedings of the 41st annual ACM symposium on Theory of computing*, *arXiv:0811.3412*, (New York, NY, USA), pp. 417–426, ACM, 2009.
- [6] D. Aharonov, I. Arad, Z. Landau, and U. Vazirani, “The 1d area law and the complexity of quantum states: A combinatorial approach,” in *FOCS '11: Proceedings of the 52nd Annual IEEE Symposium on Foundations of Computer Science (FOCS 2011)*, 2011. Accepted.