Quantum rate distortion, reverse Shannon theorems, and source-channel separation

Nilanjana Datta and Min-Hsiu Hsieh
Statistical Laboratory, University of Cambridge, Wilberforce Road, Cambridge CB3 0WB, United Kingdom

Mark M. Wilde
School of Computer Science, McGill University, Montreal, Quebec, Canada H3A 2A7
(Dated: August 25, 2011)

We derive quantum counterparts of two key theorems of classical information theory, namely, the rate distortion theorem and the source-channel separation theorem. The rate-distortion theorem gives the ultimate limits on lossy data compression, and the source-channel separation theorem implies that a two-stage protocol consisting of compression and channel coding is optimal for transmitting a memoryless source over a memoryless channel. In spite of their importance in the classical domain, there has been surprisingly little work in these areas for quantum information theory. In the present work, we prove that the quantum rate distortion function is given in terms of the regularized entanglement of purification. Although this formula is regularized, at the very least it demonstrates that Barnum's conjecture on the achievability of the coherent information for quantum rate distortion is generally false. We also determine single-letter expressions for entanglement-assisted quantum rate distortion. Moreover, we prove several quantum source-channel separation theorems. The strongest of these are in the entanglement-assisted setting, in which we establish a necessary and sufficient condition for transmitting a memoryless source over a memoryless quantum channel up to a given distortion.

Data compression is possible due to statistical redundancy in the information emitted by sources, with some signals being emitted more frequently than others. Exploiting this redundancy suitably allows one to compress data without losing essential information. If the data which is recovered after the compression-decompression process is an exact replica of the original data, then the compression is said to be *lossless*. The simplest example of an information source is a memoryless one. Such a source can be characterized by a random variable U with probability distribution $\{p_U(u)\}$ and each use of the source results in a letter u being emitted with probability $p_U(u)$. Shannon's noiseless coding theorem states that the entropy $H(U) \equiv -\sum_u p_U(u) \log_2 p_U(u)$ of such an information source is the minimum rate at which we can compress signals emitted by it [8, 19].

The requirement of a data compression scheme being lossless is often too stringent a condition, in particular for the case of multimedia data, i.e., audio, video and still images or in scenarios where insufficient storage space is available. Typically a substantial amount of data can be discarded before the information is sufficiently degraded to be noticeable. A data compression scheme is said to be *lossy* when the decompressed data is not required to be identical to the original one, but instead recovering a reasonably good approximation of the original data is considered to be good enough.

The theory of *lossy data compression*, which is also referred to as *rate distortion theory*, was developed by Shannon [5, 8, 20]. This theory deals with the trade-off between the rate of data compression and the allowed distortion. Shannon proved that, for a given memoryless information source and a distortion measure, there is a function $R(D)$, called the *rate-distortion function*, such that, if the maximum allowed distortion is D then the best possible compression rate is given by $R(D)$. He established that this rate-distortion function is equal to the minimum of the mutual information $I(U;\hat{U}) := H(U) + H(\hat{U}) - H(U,\hat{U})$ over all possible stochastic maps $p_{\hat{U}|U}(\hat{u}|u)$ that meet the distortion requirement on average:

$$R(D) = \min_{p(\hat{u}|u)\;:\;\mathbb{E}\{d(U,\hat{U})\}\leq D} I(U;\hat{U}). \tag{1}$$

In the above $d(U,\hat{U})$ denotes a suitably chosen distortion measure between the random variable U characterizing the source and the random variable $\hat{U}$ characterizing the output of the stochastic map.

Quantum rate distortion theory, that is the theory of lossy quantum data compression, was introduced by Barnum in 1998. He considered a symbol-wise entanglement fidelity as a distortion measure [2] and, with respect to it, defined the quantum rate distortion function as the minimum rate of data compression, for any given distortion. He derived a lower bound on the quantum rate distortion function, in terms of well-known entropic quantity, namely the coherent information. The latter can be viewed as one quantum analogue of mutual information, since it is known to characterize the quantum capacity of a channel [11, 15, 21], just as the mutual information characterizes the capacity of a classical channel. It is this analogy, and the fact that the classical rate distortion function is given in terms of the mutual information, that led Barnum to consider the coherent information as a candidate for the rate distortion function in the quantum realm. He also conjectured that this lower bound would be achievable.

Since Barnum's paper, there have been quite a few papers in which the problem of quantum rate distortion has either been addressed [7, 13], or mentioned in other contexts [14, 16, 17, 23]. However, not much progress has been made in proving or disproving his conjecture. In fact, in the absence of a matching upper bound, it is even unclear how good Barnum's bound is, given that the coherent information can be negative, as was pointed out in [7, 13].

In this work, we prove several important quantum rate distortion theorems and quantum source-channel separation theorems. We first consider the most natural setting for quantum rate distortion in which a compressor tries to compress a quantum information source so that a decompressor can recover it up to some distortion D according to the following distortion measure:

$$d(\rho,\mathcal{N}) = 1 - F_e(\rho,\mathcal{N}), \tag{2}$$

where F_e is the entanglement fidelity of the map $\mathcal{N}$:

$$F_e(\rho,\mathcal{N}) \equiv \langle\psi^\rho_{AA'}|(\mathrm{id}_A \otimes \mathcal{N}^{A'\to B})(\psi^\rho_{AA'})|\psi^\rho_{AA'}\rangle. \tag{3}$$

This setting is most natural whenever sufficient quantum storage is not available, but we can equivalently phrase it in a communication paradigm, where a sender has access to many uses of a noiseless qubit channel and would like to minimize the use of this resource while transmitting a quantum information source up to some distortion (see Fig. 1-(a)). We prove that the quantum rate distortion function is given in terms of a regularized entanglement of purification [22] in this case.

Theorem 1 *For a memoryless quantum information source defined by the density matrix ρ_A, and any given distortion $0 \leq D < 1$, the quantum rate distortion function is given by,*

$$R^q(D) = \lim_{k\to\infty} \frac{1}{k} \min_{\substack{\mathcal{N}^{(k)}\,: \\ d(\rho,\mathcal{N}^{(k)})\leq D}} \left[E_p^\infty(\rho^{\otimes k},\mathcal{N}^{(k)})\right], \tag{4}$$

where

$$E_p^\infty(\rho,\mathcal{N}) \equiv \lim_{n\to\infty}\frac{1}{n}E_p(\omega_{RB}^{\otimes n})$$

denotes the regularised entanglement of purification and $\omega_{RB} \equiv (\mathrm{id}_R \otimes \mathcal{N}^{A\to B})(\psi^\rho_{RA})$.

Like its classical counterpart, lossy data compression includes lossless compression as a special case. If the distortion D is set equal to zero in (4), then the state ω_{RB} becomes exactly identical to the state ψ^ρ_{RA}. Equivalently, the quantum operation $\mathcal{N}$ is given by the identity map id_A. Since the entanglement of purification is additive for tensor power states [22]:

$$E_p\big((\psi^\rho_{RA})^{\otimes n}\big) = nE_p(\psi^\rho_{RA}) = nH(\rho_A),$$

we infer that, for $D = 0$, $R^q(D)$ reduces to the von Neumann entropy of the source, which is known to be the optimal rate for lossless quantum data compression [18].

In spite of our characterization being an intractable, regularized formula, our result at the very least shows that the quantum rate distortion function is always nonnegative, demonstrating that Barnum's conjecture from Ref. [2] is false since his proposed rate-distortion function can become negative.

Our next result in quantum rate distortion is a complete characterization of the rate distortion function in an entanglement-assisted setting in a communication paradigm (see Fig. 1-(b)). The idea here is for a sender to exploit the shared entanglement and a minimal amount of classical or quantum communication in order for the receiver to recover the output of the quantum information source up to some distortion. Our main results are single-letter formulas for the entanglement-assisted rate distortion functions, in the case of noiseless classical and quantum communication, expressed in terms of a minimization of the input-output mutual information over all noisy maps that meet the distortion constraint.

Theorem 2 *For a memoryless quantum information source defined by the density matrix $\rho_{A'}$, and any given distortion $0 \leq D < 1$, the quantum rate distortion function for entanglement-assisted lossy source coding with noiseless classical communication, is given by*

$$R^q_{eac}(D) = \min_{\mathcal{N}\,:\,d(\rho,\mathcal{N})\leq D} I(A;B)_\omega\,, \tag{5}$$

where $\omega_{AB} \equiv (\mathrm{id}_A \otimes \mathcal{N}^{A'\to B})(\psi^\rho_{AA'})$, and $I(A;B)_\omega$ denotes the mutual information.

Theorem 3 *For a memoryless quantum information source defined by the density matrix $\rho_{A'}$, and any given distortion $0 \leq D < 1$, the quantum rate distortion function for entanglement-assisted lossy source coding with noiseless quantum communication, is given by*

$$R^q_{eaq}(D) = \frac{1}{2}\left[\min_{\mathcal{N}\,:\,d(\rho,\mathcal{N})\leq D} I(A;B)_\omega\right], \tag{6}$$

where $\omega_{AB} := (\mathrm{id}_A \otimes \mathcal{N}^{A'\to B})\psi^\rho_{AA'}$, and $I(A;B)_\omega$ denotes its mutual information.

It is often the case in quantum Shannon theory that the entanglement-assisted formulas end up being formally analogous to Shannon's classical formulas [4], and our result here is no exception to this trend.

Our achievability proofs of the above theorems rely on a fundamental connection between quantum reverse Shannon theorems and quantum rate-distortion protocols. In particular, if a reverse Shannon theorem is available in a given context, then it immediately leads to a rate-distortion protocol. This is done simply by choosing the simulated channel to be the one which, when acting on the source state, yields an output state which meets the distortion criterion for the desired rate-distortion task.

Note, however, that the demands of a reverse Shannon theorem are much more stringent than those of a rate-distortion protocol. A reverse Shannon theorem requires the simulation of a channel to be asymptotically

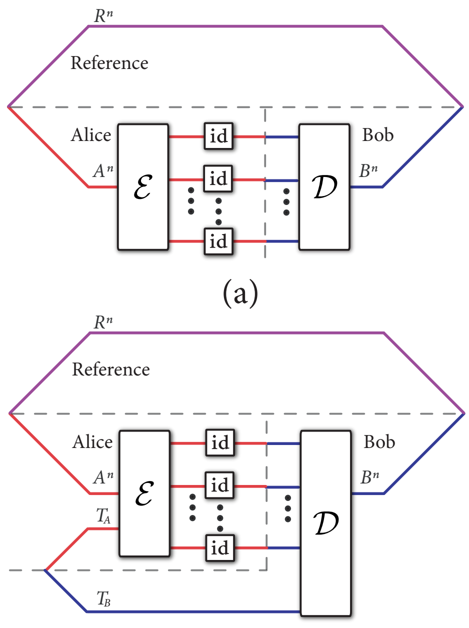

FIG. 1: The most general protocols for (a) unassisted and (b) assisted quantum rate distortion coding. In (a), Alice acts on the tensor power output of the quantum information source with a compression encoding $\mathcal{E}$. She sends the compressed qubits over noiseless quantum channels (labeled by "id") to Bob, who then performs a decompression map $\mathcal{D}$ to recover the quantum data that Alice sent. In (b), the task is similar, though this time we assume that Alice and Bob share entanglement before communication begins.

exact, whereas a rate-distortion protocol only demands that a source be reconstructed up to some average distortion constraint. The differences in these goals can impact resulting rates if sufficient correlated resources are not available [9]. On the other hand, the entanglement-assisted quantum reverse Shannon theorem suffices for producing a good entanglement-assisted rate-distortion protocol because we assume that unlimited entanglement is available in the entanglement-assisted setting.

We think that our protocol exploits more entanglement than necessary from considering what is known in the classical case regarding reverse Shannon theorems and rate-distortion coding [4, 8, 9]. First, as stated in (1), the classical mutual information minimized over all stochastic maps that meet the distortion criterion is equal to Shannon's classical rate-distortion function [8]. Bennett *et al.* have shown that the classical mutual information is also equal to the minimum rate needed to simulate a classical channel whenever free common randomness is available [4]. Thus, a simple strategy for achieving the task of rate distortion is for the parties to choose the stochastic map that minimizes the rate distortion function and simulate it with the classical reverse Shannon theorem. But this strategy uses far more classical bits than necessary whenever sufficient common randomness is not available [9]. Meanwhile, we already know that the mutual information is achievable without any common randomness if the goal is rate distortion [8].

We have further results that answer the following important communication question: what if the entropy of the source is *greater* than the capacity of the channel? For simplicity, we will only consider the entanglement-assisted setting below, but we invite the program committee to consult Ref. [10] for further source-channel separation results. Our best hope in this scenario is to allow for some distortion in the output of the source such that the rate of compression is smaller than the entropy of the source. Recall that whenever $D > 0$, the entanglement-assisted quantum rate-distortion function $R^q_{eaq}(D)$ is less than the entropy $H(R)$ of the quantum source. In this case, the condition $R^q_{eaq}(D) \leq I(\mathcal{N})/2$ is both necessary and sufficient for the reliable transmission of an information source over a noisy channel, up to some amount of distortion D, with $I(\mathcal{N})$ being given by (8) below.

Theorem 4 *The following condition is necessary and sufficient for transmitting the output of a quantum information source over an entanglement-assisted quantum channel (up to some distortion D):*

$$R^q_{eaq}(D) \leq \frac{1}{2} I(\mathcal{N}), \tag{7}$$

where $R^q_{eaq}(D)$ *is defined (6) and*

$$I(\mathcal{N}) \equiv \max_{\varphi_{AA'}} I(A;B)_\sigma, \ \sigma_{AB} \equiv \mathcal{N}^{A'\to B}(\varphi_{AA'}). \tag{8}$$

The above theorem implies that we can consider the problems of lossy data compression and channel coding separately, and the two-stage concatenation of the best lossy compression code with the best channel code is optimal.

In conclusion, we have proved several quantum rate-distortion theorems and quantum source-channel separation theorems. All of our quantum rate-distortion protocols employ the quantum reverse Shannon theorems [1, 3, 4, 6, 12]. This strategy works well whenever unlimited entanglement is available, but it clearly leads to undesirable regularized formulas in the unassisted setting. Our quantum source-channel separation theorems demonstrate in many cases that a two-stage compression-channel-coding strategy works best for memoryless sources and for quantum channels with additive capacity measures. Again, our most satisfying result is in the entanglement-assisted setting, where the pleasing result is that the entanglement-assisted rate distortion function being less than the entanglement-assisted quantum capacity is both necessary and sufficient for transmission of a source over a channel up to some distortion.

[1] Anura Abeyesinghe, Igor Devetak, Patrick Hayden, and Andreas Winter. The mother of all protocols: Restructuring quantum information's family tree. *Proceedings of the Royal Society A*, 465(2108):2537–2563, August 2009. arXiv:quant-ph/0606225.

[2] Howard Barnum. Quantum rate-distortion coding. *Physical Review A*, 62(4):042309, September 2000.

[3] Charles H. Bennett, Igor Devetak, Aram W. Harrow, Peter W. Shor, and Andreas Winter. Quantum reverse Shannon theorem. December 2009. arXiv:0912.5537.

[4] Charles H. Bennett, Peter W. Shor, John A. Smolin, and Ashish V. Thapliyal. Entanglement-assisted capacity of a quantum channel and the reverse Shannon theorem. *IEEE Transactions on Information Theory*, 48:2637–2655, 2002.

[5] Toby Berger. *Rate Distortion Theory: A Mathematical Basis for Data Compression*. Information and system sciences. Prentice Hall, 1971.

[6] Mario Berta, Matthias Christandl, and Renato Renner. The quantum reverse Shannon theorem based on one-shot information theory. December 2009. arXiv:0912.3805.

[7] Xiao-Yu Chen and Wei-Ming Wang. Entanglement information rate distortion of a quantum Gaussian source. *IEEE Transactions on Information Theory*, 54(2):743–748, February 2008.

[8] Thomas M. Cover and Joy A. Thomas. *Elements of Information Theory*. Wiley-Interscience, second edition, 2005.

[9] Paul Cuff. Communication requirements for generating correlated random variables. In *Proceedings of the 2008 International Symposium on Information Theory*, pages 1393–1397, Toronto, Ontario, Canada, July 2008. arXiv:0805.0065.

[10] Nilanjana Datta, Min-Hsiu Hsieh, and Mark M. Wilde. Quantum rate distortion, reverse Shannon theorems, and source-channel separation. August 2011. arXiv:1108.4940.

[11] Igor Devetak. The private classical capacity and quantum capacity of a quantum channel. *IEEE Transactions on Information Theory*, 51:44–55, January 2005.

[12] Igor Devetak. Triangle of dualities between quantum communication protocols. *Physical Review Letters*, 97(14):140503, October 2006.

[13] Igor Devetak and Toby Berger. Quantum rate-distortion theory for memoryless sources. *IEEE Transactions on Information Theory*, 48(6):1580–1589, June 2002. arXiv:quant-ph/0011085.

[14] Patrick Hayden, Richard Jozsa, and Andreas Winter. Trading quantum for classical resources in quantum data compression. *Journal of Mathematical Physics*, 43(9):4404–4444, September 2002. arXiv:quant-ph/0204038.

[15] Seth Lloyd. Capacity of the noisy quantum channel. *Physical Review A*, 55(3):1613–1622, March 1997.

[16] Zhicheng Luo. *Topics in quantum cryptography, quantum error correction, and channel simulation*. PhD thesis, University of Southern California, May 2009. Available from http://digitallibrary.usc.edu/.

[17] Zhicheng Luo and Igor Devetak. Channel simulation with quantum side information. *IEEE Transactions on Information Theory*, 55(3):1331–1342, March 2009. arXiv:quant-ph/0611008.

[18] Benjamin Schumacher. Quantum coding. *Physical Review A*, 51(4):2738–2747, April 1995.

[19] Claude E. Shannon. A mathematical theory of communication. *Bell System Technical Journal*, 27:379–423, 1948.

[20] Claude E. Shannon. Coding theorems for a discrete source with a fidelity criterion. *IRE International Convention Records*, 7:142–163, 1959.

[21] Peter W. Shor. The quantum channel capacity and coherent information. In *Lecture Notes, MSRI Workshop on Quantum Computation*, 2002.

[22] Barbara M. Terhal, M. Horodecki, Debbie W. Leung, and David P. DiVincenzo. The entanglement of purification. *Journal of Mathematical Physics*, 43(9):4286–4298, 2002. arXiv:quant-ph/0202044.

[23] Andreas Winter. Compression of sources of probability distributions and density operators. August 2002. arXiv:quant-ph/0208131.