Explicit lower and upper bounds on the entangled value of multiplayer XOR games

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Abstract

XOR games are the simplest model in which the nonlocal properties of entanglement manifest themselves. When there are two players, it is well known that the bias — the maximum advantage over random play — of entangled players is at most a constant times greater than that of classical players. Using tools from operator space theory, Pérez-García et al. [Comm. Math. Phys. 279 (2), 2008] showed that no such bound holds when there are three or more players: in that case the ratio of the entangled and classical biases can become unbounded and scale with the size of the game.

We give a new, simple and explicit (though still probabilistic) construction of a family of three-player XOR games for which entangled players have a large advantage over classical players. Our game has N^2 questions per player and entangled players have a factor $\tilde{\Omega}(\sqrt{N})$ advantage over classical players. Moreover, the entangled players only need to share a state of local dimension N and measure observables defined by tensor products of the Pauli matrices.

Additionally, we give the first upper bounds on the maximal violation in terms of the number of questions per player, showing that our construction is only quadratically off in that respect. Our results rely on probabilistic estimates on the norm of random matrices and higher-order tensors.

Multiplayer games, already a very successful abstraction in theoretical computer science, were first proposed as a framework in which to study the nonlocal properties of entanglement by Cleve et al. [CHTW04]. Known as *nonlocal*, or *entangled*, games, they can be thought of as an interactive re-phrasing of the familiar setting of Bell inequalities: a referee (the experimentalist) interacts with a number of players (the devices). The referee first sends a classical question (a setting) to each player. The players are all-powerful (there is no restriction on either the shared state or the measurements applied) but are not allowed to communicate: each of them must make a local measurement on his part of a shared entangled state and provide a classical answer (the outcome) to the referee's question. The referee then decides whether to accept or reject the player's answers (he evaluates the Bell functional).

In their paper, Cleve et al. gave an in-depth study of the simplest games, *two-player XOR games*.[1] Given an XOR game G, it will be convenient to measure the success of entangled (resp. classical) players through their maximum achievable *bias* $\beta^*(G)$ (resp. $\beta(G)$), defined as the maximum winning probability of entangled (resp. classical) players, *minus* the success probability for random play (each player answers a random bit, independent from his question).[2] While the CHSH example demonstrates the existence of a game for which $\beta^*(G) \geq \sqrt{2}\beta(G)$, Tsirelson [Tsi87] proved that this was close to best possible. By making a connection to the celebrated *Grothendieck inequality* he showed that for any two-player XOR game G, we have $\beta^*(G)/\beta(G) \leq K_G \lesssim 1.78$, where K_G is the Grothendieck constant.[3] We will refer to the ratio $\beta^*(G)/\beta(G)$ as the *QC gap*.

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[1] The XOR property refers to the fact that in such games each player answers with a single bit, and the referee's acceptance criterion only depends on the XOR of the bits he receives as answers. One of the most fundamental Bell inequalities, the CHSH inequality [CHSH69], fits in that framework.

[2] The motivation for subtracting the success probability of a random strategy is to provide a proper normalization for the ratios $\beta^*(G)/\beta(G)$ that will be studied below. In XOR games, random play always gives winning probability 1/2.

[3] See [BMMN11] for a recently improved upper bound on K_G.

Small QC gaps are difficult to observe in physical experiments. Since two-player XOR games cannot exhibit large QC gaps, they do not provide a reliable way to demonstrate the existence of entanglement. To achieve this, more general classes of games need to be considered, hence the natural question: For a given gap value, what is the simplest example of a game which exhibits it (if one at all exists) and what are the possible parameters (number of players, questions and answers) of such a game?

There are two directions in which one can look for generalizations. The first is to increase the number of possible answers per player. This option has so far received the most attention and has been relatively well explored [CHTW04, KRT10, JPPG+10, JP11, Reg11, BRSdW11]; there are recent explicit constructions of games which come close to achieving optimal QC gaps [BRSdW11].

The second possible avenue for generalization consists in increasing the number of players, while staying in the simple setting of binary answers and an XOR-based acceptance criterion. Our limited understanding of multipartite entanglement makes this setting more challenging, and for a long time little more than small, finite examples were known [Mer90, Zuk93]. Recently however Pérez-García et al. [PGWP+08] strikingly demonstrated that *three*-player XOR already allow for arbitrary large QC gaps, exhibiting an infinite family of such games $(G_N)_{N\in\mathbb{N}}$ for which $\beta^*(G_N)/\beta(G_N) = \tilde{\Omega}(\sqrt{N})$.

Unfortunately, the games G_N from [PGWP+08] are very large and highly non-explicit; their construction relying heavily on deep results from operator space theory it is all but impossible to even state what G_N *is*. Moreover, G_N has order 2^{N^2} questions per player, and while for one of the players the local dimension of the entangled state achieving the QC gap is only N, it is unbounded for the other two.

Our results

In this paper we give a new construction of a family of three-player XOR games for which the QC gap is unbounded. Our construction, albeit still probabilistic, is explicit and can be described in simple terms — we give it below. For a desired ratio r, our game has about r^4 questions per player, which gives an exponential improvement over the construction in [PGWP+08][4] and, as we show, is within a factor $O(r^2)$ of the smallest number possible. Moreover, to achieve such a gap entangled players only need to use Pauli observables and an entangled state of local dimension r^2 per player.

Theorem 1. *For any integer n and $N = 2^n$ there exists a three-player XOR game G_N, with N^2 questions per player, such that $\beta^*(G_N) \geq \Omega(\sqrt{N}\log^{-4} N)\,\beta(G_N)$. Moreover, there is an entangled strategy which achieves a bias of $\Omega(\sqrt{N}\log^{-4} N)\,\beta(G_N)$, uses an entangled state of local dimension N per player, and in which the players' observables are tensor products of n Pauli matrices.*

A result from [PGWP+08] shows that the dependence of the gap on the local dimension that we obtain is optimal.[5] Additionally, we prove that the dependence of the QC gap on the number of questions obtained in Theorem 1 is close to optimal.[6]

Theorem 2. *For any three-player XOR game G in which there are at most Q possible questions to the third player, we have*

$$\beta^*(G) \;\leq\; O\big(\sqrt{Q}\big)\beta(G).$$

We believe that the strength of our result rests in its accessibility and the explicitness of its parameters. The entangled strategies required in our game have a simple description and only rely on tensor products of Pauli observables, making them potentially well-suited to experiments. Moreover, our construction rests on a new connection between a specific family of XOR games and spectral properties of tensors of order 3.[7] As such, it provides a generic way to make games with large gaps from

[4]Pisier [Pis11] states that the construction in [PGWP+08] can be improved to require only r^8 questions to the first player, but still an exponential number to the other two.

[5]We give a new, simpler proof of that result in the technical paper.

[6]A similar result was recently communicated to us by Carlos Palazuelos [Pal11].

[7]A tensor of order 3, or 3-tensor, is like a matrix, but with three indices, instead of two: one can think of it as a cube of complex numbers representing a trilinear map from triples of vectors to the complex numbers.

tensors having good spectral properties, and could open the way to a *constructive* proof of the existence of a game with an unbounded QC-gap. We explain this connection in more detail next.

Proof overview and techniques

Lower bound. Our construction proceeds through two independent steps. In the first step we assume given a 3-tensor $T = T_{(i,i'),(j,j'),(k,k')}$ of dimension $N^2 \times N^2 \times N^2$, where N is a power of 2. Based on T, we define a three-player XOR game $G = G(T)$. Questions in this game are N-dimensional Pauli matrices P, Q, R, and the corresponding game coefficient[8] is defined as

$$G(P,Q,R) \;=\; \langle T, P \otimes Q \otimes R \rangle \;:=\; \sum_{(i,i'),(j,j'),(k,k')} T_{(i,i'),(j,j'),(k,k')}\, P_{i,i'} Q_{j,j'} R_{k,k'}.$$

We show that this definition results in a game whose entangled and classical biases can be directly related to *spectral properties* of the tensor T. On the one hand we show that the classical bias of the game reflects the tripartite structure of T, and is upper-bounded by the norm of T as a trilinear operator. On the other hand we show that the entangled bias is *lower-bounded* by the norm of T *as a matrix* — a bilinear operator on N^3-dimensional vectors, obtained by pairing up the indices (i,j,k) and (i',j',k'). This reduces the problem of constructing a game with large QC gap to constructing a tensor T with appropriate spectral properties.

The second step of the proof is our main technical contribution. We give a probabilistic construction of a 3-tensor T having large norm when seen as a bilinear operator (giving a large entangled bias), but low norm when seen as a trilinear operator (giving a low classical bias). To this end, we simply take T to correspond to an (almost) rank-1 matrix: letting (g_{ijk}) be a random N^3-dimensional vector with i.i.d. entries distributed as standard Gaussians, the $(i,i'),(j,j'),(k,k')$-th entry of T is $g_{ijk}\, g_{i'j'k'}$ if $i \neq i'$, $j \neq j'$ and $k \neq k'$, and 0 otherwise. The fact that T, when seen as a matrix, is close to having rank 1 makes it easy to lower bound its spectral norm and thereby lower bound the entangled bias of the corresponding game. We are then able to upper bound the norm of T as a trilinear operator by a standard, though delicate, ε-net argument.

Upper bound. The main ingredients for our upper bound on the QC gap in terms of the number of questions are a simple but useful "decoupling" technique due to Paulsen [Pau92] and Grothendieck's inequality. Paulsen's technique uses the simple fact that for i.i.d. symmetrically distributed Bernoulli random variables $\sigma_1, \ldots, \sigma_Q$, we have $\mathbb{E}[\sigma_k \sigma_\ell] = \delta_{k,\ell}$. For some matrices $A_1, \ldots, A_Q$ and $B_1, \ldots, B_Q$, a sum of the form $\sum_{k=1}^{Q} A_k \otimes B_k$ can then be written as $\mathbb{E}\big[\big(\sum_{k=1}^{Q} \sigma_k A_k\big) \otimes \big(\sum_{\ell=1}^{Q} \sigma_\ell B_\ell\big)\big]$, thereby decoupling the B_ks from the A_ks. In the context of three-player XOR games, the B_k should be thought of as the third players' observables and the A_k as a conglomerate of the game tensor and the first two players' observables. Using this, we can turn the third entangled player into a classical player at a cost of a factor $\sqrt{Q}$ in the overall bias. Being left with only two entangled players, we can finish with a straightforward application of Grothendieck's inequality, as was done previously by Tsirelson [Tsi87].

Generalizations. Our results all have natural extensions to an arbitrary number of players. In particular, we can show that the following holds, for any $K \geq 3$:

1. For any integer N that is a power of 2, there exists a K-player XOR game G, with N^2 questions per player, such that $\beta^*(G) \geq \Omega\big((N \log^{-8} N)^{(K-2)/2}\big)\beta(G)$, and there is a quantum strategy achieving this gap and using only N-dimensional Pauli matrices.
2. If G is a K-player XOR game in which at least $K-2$ of the players have at most Q possible questions each, then $\beta^*(G) \leq O(Q^{(K-2)/2})\beta(G)$.
3. If G is a K-player XOR game in which the shared state of the players is restricted to have local dimension d on at least $K-2$ of the players, then $\beta^*(G) \leq O\big((d \log d)^{(K-2)/2}\big)\beta(G)$.

[8]The equation below defines a complex number. Taking its real or imaginary part would result in a Bell functional, which can in turn easily be transformed into an XOR game through a proper normalization. Details are given in the full paper.

References

[BMMN11] M. Braverman, K. Makarychev, Y. Makarychev, and A. Naor. The grothendieck constant is strictly smaller than krivine's bound. arXiv:1103.6161, to appear in Proc. FOCS 2011, 2011.

[BRSdW11] H. Buhrman, O. Regev, G. Scarpa, and R. de Wolf. Near-optimal and explicit bell inequality violations. arXiv:1012.5043, to appear in Proc. CCC 2011, 2011.

[CHSH69] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt. Proposed experiment to test local hidden-variable theories. *Phys. Rev. Lett.*, 23:880–884, 1969.

[CHTW04] R. Cleve, P. Høyer, B. Toner, and J. Watrous. Consequences and limits of nonlocal strategies. In *Proceedings of the 19th IEEE Conference on Computational Complexity (CCC 2004)*, pages 236–249, 2004. quant-ph/0404076.

[JP11] M. Junge and C. Palazuelos. Large violation of bell inequalities with low entanglement. *Communications in Mathematical Physics*, pages 1–52, 2011. 10.1007/s00220-011-1296-8.

[JPPG$^+$10] M. Junge, C. Palazuelos, D. Prez-Garca, I. Villanueva, and M. Wolf. Unbounded violations of bipartite bell inequalities via operator space theory. *Communications in Mathematical Physics*, 300:715–739, 2010. 10.1007/s00220-010-1125-5.

[KRT10] J. Kempe, O. Regev, and B. Toner. Unique games with entangled provers are easy. *SIAM J. Comput.*, 39:3207–3229, July 2010.

[Mer90] N. D. Mermin. Extreme quantum entanglement in a superposition of macroscopically distinct states. *Phys. Rev. Lett.*, 65(15):1838–1840, Oct 1990.

[Pal11] C. Palazuelos. Personal communication, 2011.

[Pau92] V. I. Paulsen. Representations of function algebras, abstract operator spaces, and Banach space geometry. *Journal of Functional Analysis*, 109(1):113 – 129, 1992.

[PGWP$^+$08] D. Pérez-García, M. M. Wolf, C. Palazuelos, I. Villanueva, and M. Junge. Unbounded violation of tripartite bell inequalities. *Communications in Mathematical Physics*, 279:455, 2008.

[Pis11] G. Pisier. Grothendieck's Theorem, past and present. *Arxiv preprint arXiv:1101.4195*, 2011.

[Reg11] O. Regev. Bell violations through independent bases games. Technical report arXiv:1001.0576, 2011.

[Tsi87] B. S. Tsirelson. Quantum analogues of the Bell inequalities. The case of two spatially separated domains. *J. Soviet Math.*, 36:557–570, 1987.

[Zuk93] M. Zukowski. Bell theorem involving all settings of measuring apparatus. *Physics Letters A*, 177(4-5):290 – 296, 1993.