Local stabilizer codes in three dimensions without string logical operators

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Encoding a quantum state into a many-body state drew attention in view of fault-tolerant quantum computation. Topologically ordered system naturally makes the connection precise since such a system by definition has a ground state subspace that does not allow any local observable to resolve internal structure. Any local operation or noise is not capable of altering a state in the topologically ordered subspace. Thus, one would hope to find a noise-free quantum memory from a many-body system.

The topological order, however, is fragile against thermal noise that is ever present in every physical system. An intuition can be obtained by considering a simple model, Kitaev toric code. In this model, electric or magnetic charges are created in pairs in contact with a noisy environment and can propagate through the system without any barrier. At a non-zero temperature, there is non-zero density of charges whose positions fluctuate. They often meet their anti-particle partners to annihilate and are created with nonzero probability. When the overall path of charges form a topologically nontrivial loop, it is as if an nontrivial operator has acted on the topologically ordered subspace. This action occurs frequently and the protected ground space at zero temperature is no more protected. Note that the charges are the end points of a string operator.

Two solutions to this issue can be given. One is simply to make the energy of the `charge' macroscopic as in the four dimensional toric code. Another is to introduce a barrier for the propagating charge. But, all previous models in two or three dimensions, at least those based on stabilizer codes, had string-like operators that convey charges without any barrier. For this reason, they all are not self-correcting. (With disordered Hamiltonian, it is pointed out that there could be a barrier. We focus on clean systems.)

In this work, we find new three dimensional models based on stabilizer codes (a.k.a., quantum additive codes) that prohibit any propagation of charges. In particular, we show that there are no string-like logical operators. Our models have strictly local interaction with bounded strength, obviously exactly solvable energy spectrum, and topologically ordered ground state.

We start by considering general local stabilizer codes. Namely, we consider simple cubic lattice with periodic boundary conditions with possibly many qubits per site. The interaction terms are supported on eight sites that form a unit cube; each term is identified with a generator of stabilizer group. The Hamiltonian is the negative sum of all stabilizer generators. We impose conditions on the generators such that it does not allow any nontrivial logical operators along straight lines. We further assume codes be translation-invariant and of CSS-type with one X- and one Z-type generators for each unit cube. The formalism of stabilizer codes gives very handy description of the conditions in terms of linear algebra over the binary field. Solving linear equations by brute-force, we find 17 codes.

In order to prove that a code is free of string logical operators, a precise meaning of strings must be defined. If it were a continuum theory, there wouldn't be a problem of calling an object a string. In discrete lattice, however, it is not obvious. Intuitively, a string in a lattice is an infinite object stretched along some direction, or an infinite family of finite ones that share a certain direction. In order to avoid potential issues with the infinity, we introduce a finite object, string segment. For clarity, we define it only for translation-invariant stabilizer codes: A logical string segment consists of a finitely supported Pauli operator and two disjoint regions, called anchors, that contain all excitations; any excitation is allowed to be only inside the anchors. That is, a logical string segment is almost a logical operator except for the anchors. An identity operator with arbitrary anchors is trivially a logical string segment. Hence, we call a logical string segment nontrivial if any equivalent (up to stabilizers) logical string segment `connects' anchors. Any stabilizer code with string logical operator in a common sense, has property that the distance between anchors of nontrivial logical string segments is unbounded. In contrast, 2D Ising model or 4D toric code has property that nontrivial logical string segment is short and the distance between anchors is bounded.

We prove that four codes among 17 have nontrivial logical string segments whose length is bounded. The proof consists of two steps. First, we show any logical string segment is equivalent to a juxtaposition of three `flat' segments that are parallel to the principal axes of the cubic lattice. Second, imposing consistency conditions on the configuration of single qubit Pauli operators that form the flat segment, we show the segment cannot be too long. Again, the stabilizer formalism makes the calculation clear and easy. An interpretation of this proof is that two regions of charges that are far apart are only trivial charges; they are created locally. Said differently, there is no Pauli operator capable of moving nontrivial charges to a distant location. If a nontrivial charge is to be moved to somewhere, it must create another charge at a yet different place.

The four models are all topologically ordered. This is proved by showing that if a finite support Pauli operator commutes with all stabilizers, then it must also be a stabilizer. The proof is a variant of the first step of the no-string proof. Given this, together with the fact that the model encodes at least one logical qubit, one sees that the codes have code distance larger than the linear system size in the periodic boundary conditions. That is, there is no local observable to distinguish ground states.

A totally unexpected result is that the code space dimension or the logarithm of the ground state degeneracy is highly sensitive to the system size. It gets bigger as there are more factors of 2 in the prime decomposition of the system size, but sometimes drops to 2 if linear system size increases by 1.

Our models open a possibility of self-correcting quantum memory in three dimensions. They disprove a conjecture that there is a dimensional duality between logical operators for geometrically local quantum codes. The models display true quantum glassiness; not just some kind of charge is immobile, but all kinds of charge are immobile in the low temperature limit. It is also illustrated that the local indistinguishability of the ground states does not imply that the degeneracy only depend on the topology of the space into which the lattice is embedded.