Infinitely many constrained inequalities for the von Neumann entropy

Josh Cadney,[1,*] Noah Linden,[1] and Andreas Winter[1,2]
[1] *Department of Mathematics, University of Bristol, Bristol BS8 1TW, U.K.*
[2] *Centre for Quantum Technologies, National University of Singapore, Singapore 117542*
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We exhibit infinitely many new, constrained inequalities for the von Neumann entropy, and show that they are independent of each other and the known inequalities obeyed by the von Neumann entropy (basically strong subadditivity). The new inequalities were proved originally by Makarychev *et al.* [Commun. Inf. Syst., 2(2):147-166, 2002] for the Shannon entropy, using properties of probability distributions. Our approach extends the proof of the inequalities to the quantum domain, and includes their independence for the quantum and also the classical cases.

See http://arxiv.org/PS_cache/arxiv/pdf/1107/1107.0624v1.pdf for a full technical version.

The von Neumann entropy, given by $S(\rho) = -\mathrm{Tr}\rho\log\rho$ for a quantum state (density operator) ρ, is one of the cornerstones of quantum information theory. It plays an essential role in the expressions for the best achievable rates of virtually every coding theorem. In particular, when proving the optimality of these expressions, it is the inequalities governing the relative magnitudes of the entropies of different subsystems which are important. There are essentially two such inequalities known, the so called basic inequalities:

$$I(A:B|C) := -S(C) + S(AC) + S(BC) - S(ABC) \geq 0, \tag{SSA}$$

$$S(AB) + S(AC) - S(B) - S(C) \geq 0. \tag{WMO}$$

Inequality (SSA) is known as *strong subadditivity* and was proved by Lieb and Ruskai [1] and the expression on the left hand side as the (quantum) conditional mutual information; inequality (WMO) is usually called *weak monotonicity.*

Here we examine an alternate perspective on the entropy, which we explain now. Given a multipartite state ρ on a set of parties (quantum systems) $N = \{X_1, \ldots, X_n\}$, we define the *entropy vector* of ρ to be the vector in $2^n - 1$ dimensional real space, which is a list of the entropies of all the various subsets of N. For example, if ρ is a bipartite state, then the entropy vector of ρ is $(S(\rho_{X_1}), S(\rho_{X_2}), S(\rho_{X_1X_2}))$. It is then natural to ask the question: which vectors can arise as the entropies of quantum states? For example, the vector $(1, 1, 2)$ is the entropy vector of the maximally mixed state on two spin-$\frac{1}{2}$ systems. However, we know that the vector $(1, 1, 3)$ cannot represent the entropies of any quantum state since in general the quantity $S(X_1)_\rho + S(X_2)_\rho - S(X_1X_2)_\rho$ is non-negative, whereas here it is equal to -1. Thus, the question of which vectors can be realized by quantum states is inextricably linked to the knowledge of entropy inequalities.

The region of points in $\mathbb{R}^{2^n-1}$ which satisfy all instances of (SSA) and (WMO) forms a polyhedral cone, which we denote Σ_n, which must contain all entropy vectors. For $n = 2$ and $n = 3$ it has been shown [2] that the closure of the set of entropy vectors, which we will denote $\overline{\Sigma}^*_n$, is exactly Σ_n. In other words, a vector can be realized, with arbitrary accuracy, as the entropy vector of a quantum state if and only if it satisfies all the basic inequalities. For $n \geq 4$ it can again be shown that $\overline{\Sigma}^*_n$ forms a convex cone, however, it is unknown whether or not this cone is the same as that which is determined by the basic inequalities.

*Electronic address: `josh.cadney@bristol.ac.uk`

In classical information theory, the Shannon entropy of a random variable, given by $H(X) = -\sum_{x\in\mathcal{X}} p_X(x)\log p_X(x)$, plays an analogous role to the von Neumann entropy. It satisfies the same basic inequalities as above, with (WMO) replaced by the stronger condition of *monotonicity*: $H(AB) \geq H(A)$. The analogous classical problem to the one we study here has been extensively studied for quite some time. First Zhang and Yeung [3], and then Makarychev *et al.* [4] and Dougherty *et al.* [5] found new inequalities, which are not implied by the basic inequalities. Matúš [6] even proved that for $n \geq 4$ the classical entropy cone is not polyhedral, i.e. it cannot be described by any finite set of linear inequalities.

In the quantum case, only one inequality is known which cannot be deduced from the basic inequalities [7], and it is a so-called constrained inequality – an inequality which holds whenever certain conditional mutual informations are zero. This shows that parts of certain faces of the cone Σ_n do not contain any entropy vectors of quantum states. This is not enough, however, to conclude that the entropy cone, $\overline{\Sigma}_n^*$, is strictly smaller, as we are concerned with the closure. In fact, it remains a major open problem to decide the existence of an unconstrained inequality for the von Neumann entropy that is not implied by the basic inequalities.

Our main result is the following theorem, whose analogue was proved in [4] for the Shannon entropy:

Theorem 1. *Let ρ be a multipartite quantum state on parties $\{A, B, C, X_1, \ldots, X_n\}$ which satisfies the constraints:*

$$I(A:C|B)_\rho = I(B:C|A)_\rho = 0. \tag{1}$$

Then the following inequality holds:

$$S(X_1 \ldots X_n)_\rho + (n-1)I(AB:C)_\rho \leq \sum_{i=1}^{n} S(X_i)_\rho + \sum_{i=1}^{n} I(A:B|X_i)_\rho. \tag{2}$$

Notice that this theorem describes an infinite family of inequalities, one for each value of n. In the case $n = 1$, (2) reads:

$$I(A:B|X_1) \geq 0, \tag{3}$$

which is simply one of the basic inequalities. With this in mind, one may suspect that all these inequalities can be deduced from the basic inequalities. However, we show that this is not the case; for $n \geq 2$ the inequality (2) cannot be deduced from (SSA) and (WMO), nor also from any combination of inequality (2) for any other values of n.

Theorem 1 gives an infinite family of independent inequalities that hold for the von Neumann entropy. However, since these inequalities are constrained they only reveal information about the boundary of Σ_n^*, and so they are still not enough to conclude that $\overline{\Sigma_n^*} \subsetneq \Sigma_n$.

Towards this end, we note that the inequalities we proved are the same as those proved in section 3 of [4] for the Shannon entropy, and and indeed our proof follows a similar outline. In [4] a similar family of *unconstrained* inequalities for the Shannon entropy are also proved, using a method which, in some sense, generalises the constrained proof. It may be possible that this proof can be generalised to apply to the von Neumann entropy, however, it seems as though some new tools would have to be developed first.

In any case, we believe it possible that a deeper connection between classical and quantum entropy inequalities exists. We have tested many of the new non-Shannon type inequalities on quantum states, using a numerical optimisation program, and have not been able to find a single violation (although limits on processing power restrict us to Hilbert spaces with small local dimensions). We also note that all these new inequalities are balanced, and that the only entropy inequality which is known to be true in the classical but not in the quantum case, is monotonicity – which is unbalanced.

We therefore are led to speculate whether, in fact, *all* balanced inequalities that hold for the Shannon entropy also hold for the von Neumann entropy.

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