Bounds on the distance between a unital quantum channel and the convex hull of unitary channels, with applications to the asymptotic quantum Birkhoff conjecture

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Abstract. Motivated by the recent resolution of Asymptotic Quantum Birkhoff Conjecture (AQBC), we attempt to estimate the distance between a given unital quantum channel and the convex hull of unitary channels. We provide two lower bounds on this distance by employing techniques from quantum information and operator algebra, respectively. We then show how to apply these results to construct some explicit counterexamples to AQBC.

Keywords: unital quantum channels, convex hull of unitary channels, asymptotic quantum Birkhoff Conjecture, Diamond norm, Schur channel

1. Introduction

The quantum Birkhoff conjecure, originated from the Birkhoff's celebrated characterization of the extreme points of doubly stochastic matrices, was to ask whether one can always decompose a unital quantum channel $\Phi \in \mathrm{T}(\mathcal{H}_d)$ into a mixture of unitary channels from $\mathbb{U}(\mathcal{H}_d)$. Unfortunately, this conjecture is only true for $d \leq 2$, and counterexamples exist whenever $d \geq 3$ [1, 2, 3]. This suggests the following quantity to measure the distance between Φ and the convex hull of unitary channels.

$$D(\Phi, \mathrm{Conv}(\mathbb{U}(\mathcal{H}))) = \sup\{D(\Phi, \Psi) : \Psi \in \mathrm{Conv}(\mathbb{U}(\mathcal{H}))\},$$

where $D(\Phi, \Psi)$ will be given by the diamond norm of $\Phi - \Psi$.

Motivated by some results in about the environment-assisted quantum capacity and in an attempt to remedy the conjecture in certain way, Smolin, Verstraete, and Winter proposed the following

Conjecture 1 *(Asymptotic Quantum Birkhoff Conjecture [4]) Let $\Phi \in \mathrm{T}(\mathcal{H})$ be a unital channel, then $\Phi^{\otimes n}$ can be approximated by a mixture of unitary channels from $\mathbb{U}(\mathcal{H}^{\otimes n})$ with arbitrary precision. That is*

$$\lim_{n\to\infty} D(\Phi^{\otimes n}, \mathrm{Conv}(\mathbb{U}(\mathcal{H}^{\otimes n}))) = 0.$$

Due to its significance, the asymptotic quantum Birkhoff conjecture was listed as one of major open problems in quantum information theory [5]. Mendl and Wolf's work further supported this conjecture by presenting a unital channel Φ such that $\Phi^{\otimes 2}$ is a mixture of unitary channels although Φ itself is not [6]. Recently Haagerup and Musat disproved this asymptotic version by exhibiting a class of so-called non-factorizable maps as counterexamples [7]. Shor and cowokers also showed that a slight variant of the counterexample to quantum Birkhoff conjecture presented in Ref. [1] remains a counterexample to the asymptotic case [8]. The interesting thing here is that all these counterexamples are non-factorizable maps, and it was unknown whether any facterizable map will fulfill AQBC.

Motivated by these progresses and in order to better understand the structure of unital channels, in this abstract we are interested in estimating the trace distance between a unital quantum channel and the convex hull of unitary channels, say $D(\Phi, \mathrm{Conv}(\mathrm{U}(\mathcal{H})))$. We find that this distance is interesting even from the perspective of quantum channel discrimination: Suppose we are given an unknown quantum channel, which is secretly chosen between Φ and some $\Psi \in \mathrm{Conv}(\mathrm{U}(\mathcal{H})))$ with equal probability 1/2. Then due to the operational meaning of trace distance, we conclude that the success probability of discrimination is upper bounded by $1/2 + 1/4D(\Phi, \mathrm{Conv}(\mathrm{U}(\mathcal{H})))$, and is strictly larger than 1/2 whenever Φ is not a mixture of unitary channels. In the following discussion a quantum channel Φ over linear operator space $\mathrm{L}(\mathcal{H}_d)$ is said to be a Schur channel if Φ can be written as $\Phi = \sum_k E_k \cdot E_k^\dagger$ with each diagonal E_k and $\sum_k E_k^\dagger E_k = I_d$. The class of all Schur channels over $\mathrm{L}(\mathcal{H}_d)$ is denoted as $\mathrm{S}(\mathcal{H}_d)$ or simply S_d. We should point out that all technical proofs have been omitted here and we recommend the interested readers to read the attached full paper for details.

2. A Lower bound for the distance between a quantum channel and the convex hull of unitary channels

Our first result is a computable lower bound for $D(\Phi, \mathrm{Conv}(\mathbb{U}(\mathcal{H})))$ when the Kraus operator space of Φ does not contain any unitary operator. (Note that the Kraus operator space $K(\Phi) = \mathrm{span}\{E_k\}$ for a quantum channel $\Phi = \sum_k E_k \cdot E_k^\dagger$). This enables us to derive many counterexamples for AQBC, including some factorizable maps presented in Ref. [7]. It is worth pointing out that this proof employs only some basic techniques from quantum information theory.

Lemma 1 *For any quantum channel $\Phi \in \mathrm{T}(\mathcal{H}_d)$ such*

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that $K(\Phi) \cap \mathrm{U}(\mathcal{H}_d) = \emptyset$, we have

$$D(\Phi, \mathrm{Conv}(\mathbb{U}(\mathcal{H}_d))) \geq \min_{L \in K(\Phi)} \frac{\mathrm{Tr}(|L| - I_d)^2}{d} = C_\Phi > 0.$$

Theorem 2 *Let $\Phi \in \mathrm{T}(\mathcal{H}_d)$ be a quantum channel, and let $\Psi \in \mathrm{S}(\mathcal{H}_m)$ be any Schur channel. Then*

$$D(\Psi \otimes \Phi, \mathrm{Conv}(\mathbb{U}(\mathcal{H}_m \otimes \mathcal{H}_d))) \geq C_\Phi.$$

As a direct corollary, we have the following

Corollary 3 *For any Schur channel $\Phi \in \mathrm{S}(\mathcal{H}_d)$, if $K(\Phi) \cap \mathrm{U}(\mathcal{H}_d) = \emptyset$, then*

$$D(\Phi^{\otimes n}, \mathrm{Conv}(\mathbb{U}(\mathcal{H}_d^{\otimes n}))) \geq C_\Phi > 0, \ \forall n \geq 1.$$

The above results allow us to construct some counterexamples to AQBC easily. Our basic strategy is to construct Schur channel Φ satisfying $K(\Phi) \cap \mathrm{U}(\mathcal{H}) = \emptyset$. Then the statement that Φ is a counterexample to AQBC follows directly from Corollary 3.

Example 1: ([3] (Section 4.3)). $\Phi = E_1 \cdot E_1^\dagger + E_2 \cdot E_2^\dagger$, where

$$E_1 = \mathrm{Diag}(1, 0, 1/\sqrt{2}, 1/\sqrt{2}), \ E_2 = \mathrm{Diag}(0, 1, 1/\sqrt{2}, -i/\sqrt{2}).$$

Example 2: ([7], Example 3.3) $\Phi = \sum_{k=1}^3 E_k \cdot E_k^\dagger$, where

$$E_1 = \mathrm{Diag}(1, 1/\sqrt{5}I), E_2 = \mathrm{Diag}(0, 2/\sqrt{5}Z), E_3 = E_2^\dagger,$$

where $Z = \mathrm{Diag}(1, 2\pi i/5, 4\pi i/5, 6\pi i/5, 8\pi i/5)$ satisfying $Z^5 = I$. As shown in [7], one can choose a set of Hermitian Kraus operators F_1, F_2, F_3 such that

$$F_1 = E_1, \ F_2 = 1/2(E_2 + E_3), \ F_3 = 1/2i(E_2 - E_3).$$

It is easy to see that $\Phi = \sum_{k=1}^3 F_k \cdot F_k^\dagger$. By Corollary 2.5 of Ref. [7], Φ is a factorizable map. This gives us a factorizable map which is also a counterexample to AQBC.

3. Bounds on the distance between a Schur channel and the convex hull of unitary channels

We now go further to study the class of Schur channels. In this case we are able to show that up to a factor of 1/2, any Schur channel can be approximated by a mixture of diagonal unitary channels. We will denote the set of Schur channels that are also mixtures of diagonal unitary channels by $\Lambda(\mathcal{H}_d)$.

Theorem 4 *For given Schur channel $\Phi \in \mathrm{S}(\mathcal{H}_d)$, we have*

$$\frac{1}{2} D(\Phi, \Lambda(\mathcal{H}_d)) \leq D(\Phi, \mathrm{Conv}(\mathbb{U}(\mathcal{H}_d))) \leq D(\Phi, \Lambda(\mathcal{H}_d)).$$

Theorem 5 *For given Schur channel $\Phi \in \mathrm{S}(\mathcal{H}_d)$ and arbitrary $\Psi \in \mathrm{S}(\mathcal{H}_m)$, we have*

$$D(\Psi \otimes \Phi, \Lambda(\mathcal{H}_{m \otimes d})) \geq D(\Phi, \Lambda(\mathcal{H}_d)), \forall \Psi \in \mathrm{S}(\mathcal{H}_m). \quad (1)$$

A somewhat surprising fact is that the above two results can be used to derive some results first obtained by Haagerup and Musat in Ref. [9].

Theorem 6 *(Haagerup and Musat [9]) For given Schur channel $\Phi \in \mathrm{S}(\mathcal{H}_d)$ and arbitrary $\Psi \in \mathrm{S}(\mathcal{H}_m)$, we have*

$$D(\Psi \otimes \Phi, \mathrm{Conv}(\mathbb{U}_{m \otimes d})) \geq \frac{1}{2} D(\Phi, \mathrm{Conv}(\mathbb{U}(\mathcal{H}_d))). \quad (2)$$

Corollary 7 *((Haagerup and Musat [9]) Let $\Phi \in \mathrm{S}(\mathcal{H}_d)$ be a Schur channel that does not satisfy the quantum Birkhoff property, that is, $\Phi \notin \mathrm{Conv}(\mathbb{U}(\mathcal{H}_d))$. Then Φ does not satisfy the asymptotic quantum Birkhoff property, and*

$$D(\Phi^{\otimes n}, \mathrm{Conv}(\mathbb{U}(\mathcal{H}_d^{\otimes n}))) \geq \frac{1}{2} D(\Phi, \mathrm{Conv}(\mathbb{U}(\mathcal{H}_d))).$$

4. Remarks and acknowledgements.

After we obtained the results in Section 2, and were working on the proof of the Theorem 4 in Section 3, the second author R.D. happened to learn from Prof. M. B. Ruskai that Haagerup and Musat had made further progress on the connection between Schur channels and AQBC. Namely, they obtained Theorem 6 and thus showed that any Schur channel that violates QBC (including some factorizable maps) should also be a counterexample to AQBC [9]. They also provided a modified version of the connection between factorizable maps satisfying AQBC and Connes embedding conjecture. The proof of Theorem 4 has employed some similar techniques in [9].

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