Quantum simulation of time-dependent Hamiltonians and the convenient illusion of Hilbert space

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We consider the manifold of all quantum many-body states that can be generated by arbitrary time-dependent local Hamiltonians in a time that scales polynomially in the system size, and show that it occupies an exponentially small volume in Hilbert space. This implies that the overwhelming majority of states in Hilbert space are not physical as they can only be produced after an exponentially long time. We establish this fact by making use of a time-dependent generalization of the Suzuki-Trotter expansion, followed by a counting argument. This also demonstrates that a computational model based on arbitrarily rapidly changing Hamiltonians is no more powerful than the standard quantum circuit model.

I. INTRODUCTION AND SUMMARY OF RESULTS

The Hilbert space of a quantum system is big—its dimension grows exponentially with the number of particles it contains. Thus, parametrizing a generic quantum state of N particles requires an exponential number of real parameters. Fortunately, the states of many physical systems of interest appear to occupy a tiny sub-manifold of this gigantic space. Indeed, the essential physical features of many systems can be explained by variational states specified with a small number of parameters. Well-known examples include the BCS state for superconductivity [1], Laughlin's state for fractional quantum Hall liquids [2], tensor network states occurring in DMRG and real-space renormalization methods [3–6]. In these cases, the states are described by number of parameters scales only polynomially with N.

In this paper, we attempt to define the class of *physical* states of a many-body quantum system with local Hilbert spaces of bounded dimensions, and prove that they represent an exponentially small sub-manifold of the Hilbert space. We say that a state is physical if it can be reached, starting in some fiducial state (e.g. a ferromagnetic state, or the vacuum), by an evolution generated by any time-dependent quantum many-body Hammiltonian, with the constraint that 1) the Hamiltonian is local in the sense that it is the sum of terms each acting on at most k bodies for some constant k independent of N and 2) the duration of the evolution scales at most as a polynomial in the number of particles in the system. The assumption about the initial fiducial state is artificial; we could alternatively define the class of physical evolutions for quantum many-body systems as the ones generated by Hamiltonians obeying constraints 1 and 2, and would reach the same conclusions.

Constraint 2 is very much reminiscent of the way complexity classes are defined in theoretical computer science, where the central object of study is the scaling of the time required to solve a problem as a function of its input size. The classical analogue for our problem is a well-known counting argument of Shannon [7] demonstrating that the number of boolean functions of N bits is doubly exponentially in N (i.e., 2^{2^N}), with the consequence that no efficient (i.e. polynomial) algorithm can exist to compute the overwhelming majority of those functions. Indeed, the number of different functions that can be encoded by all classical circuits of polynomial depth scales as $2^{poly(N)}$, which is exponentially smaller than the total number of Boolean functions.

Our contribution is a quantum generalization of this result, which has in part been a folklore theorem in the quantum information community for some time. The crux of our argument is to demonstrate that the dynamics generated by any local Hamiltonian, without any assumptions on its time-dependence, can be simulated by a quantum circuit of polynomial size. This is an important result: previously-known Hamiltonian-simulation methods [8–13] produced circuits of size depending on the rate of change of the Hamiltonian, scaling e.g. with $\|\partial H/\partial t\|$ or some higher order derivatives. From our result we can count the number of physical states (or physical evolution operators) by reproducing the folk theorem and show that they occupy an exponentially small fraction of Hilbert space. Note that a direct parameter counting would not produce this result because we impose no restriction on the time-dependence of the Hamiltonian. The complete description of a rapidly changing Hamiltonian requires lots of information and, from this perspective, there are in principle enough parameters to reach all states in the Hilbert space. This leads to the conclusion that most states in the Hilbert space are not physical and they can only be reached after an exponentially long time. This has to be contrasted to the classical case, where all *states* of N bits correspond to physical states: they can easily be generated by trivial depth-one circuits. The difference between classical and quantum behavior is due to the existence of quantum entanglement.

Lastly, our result can be seen as an illustration of the decoupling principle which states that the high-frequency fluctuations of the Hamiltonian should not affect the low-energy physics. As a consequence, it should possible to largely ignore these fluctuations without significantly modifying the dynamics of the system. This is the working principle behind renormalization group methods of quantum field theory and quantum many-body physics. Indeed, our analysis leads to a rigorous demonstration of this important principle for local Hamiltonians encountered in quantum many-body physics.

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