Constructing Quantum Network Coding Schemes from Classical Nonlinear Protocols

Hirotada Kobayashi
National Institute of Informatics

François Le Gall
The University of Tokyo

Harumichi Nishimura
Osaka Prefecture University

Martin Roetteler
NEC America Labs

QIP'11 - 13 January 2011

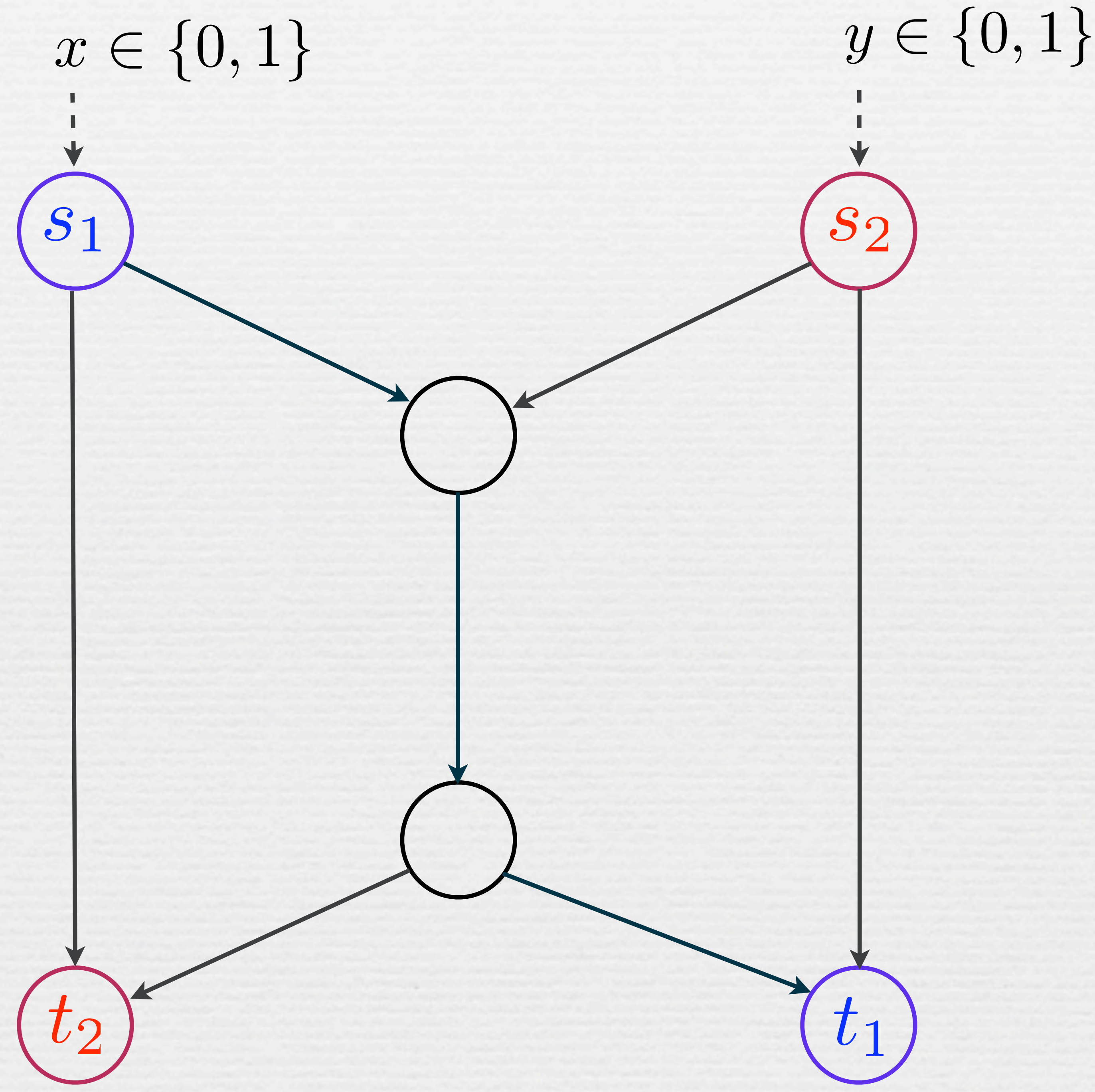
Classical Network Coding: Butterfly Graph

- two sources s_1 and s_2
- two targets t_1 and t_2

Goal:

$$\boxed{\begin{array}{l}\text{send } x \text{ to } t_1 \\ \text{send } y \text{ to } t_2\end{array}}$$

- each edge has capacity 1 bit



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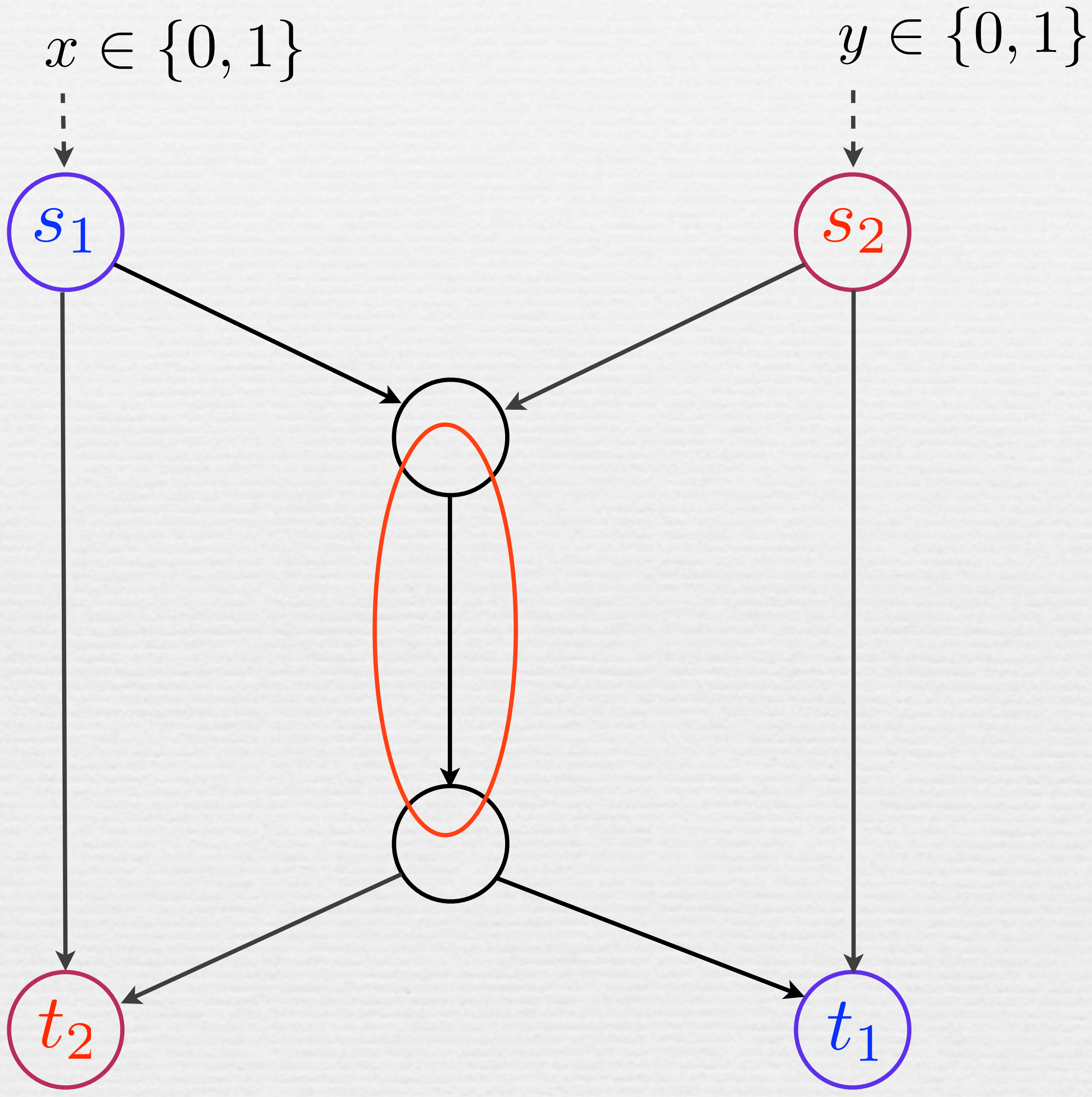

routing cannot be used

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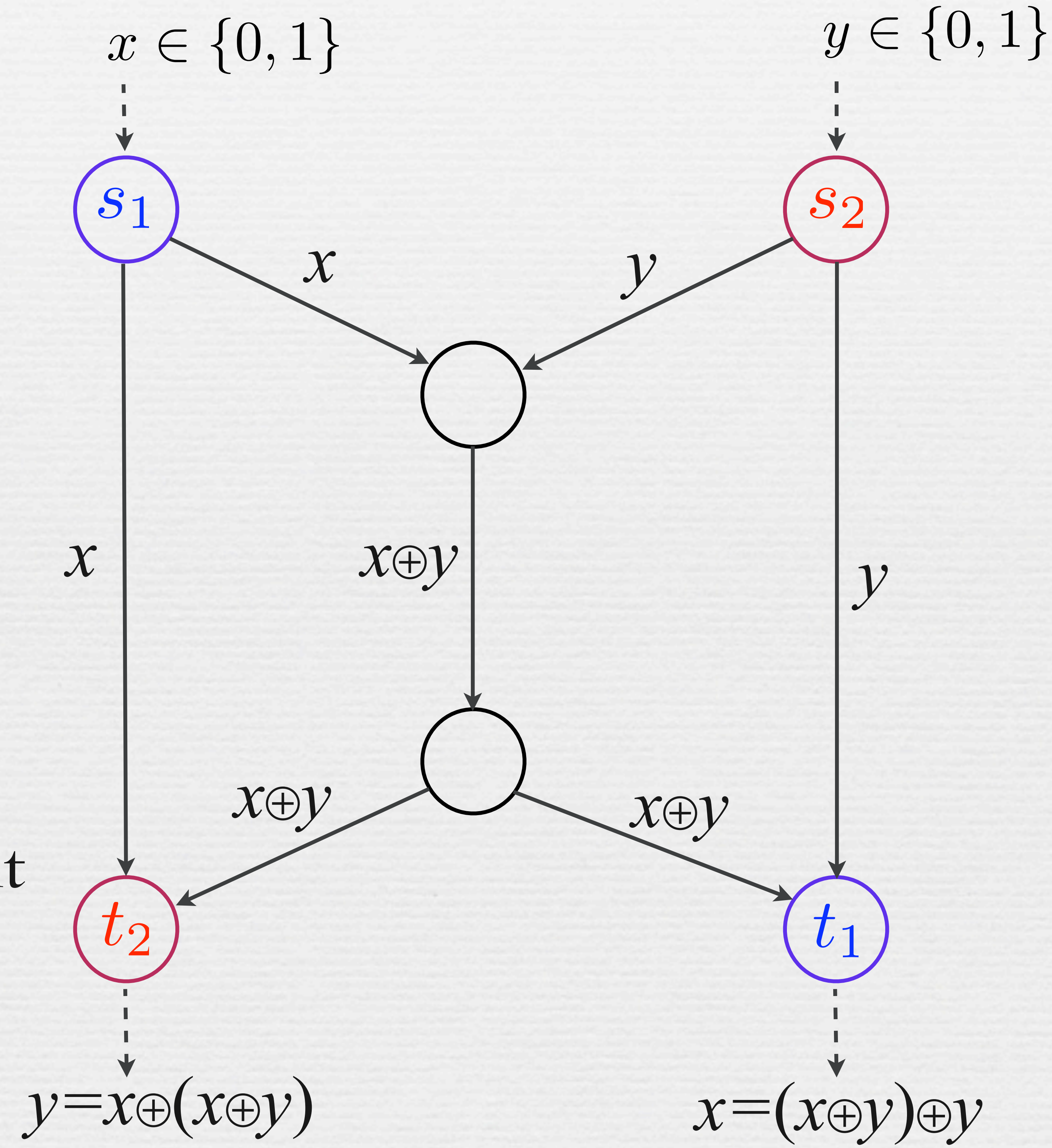

Network Coding

[Ahlswede, Cai, Li, Yeung, 00]

Quantum Network Coding

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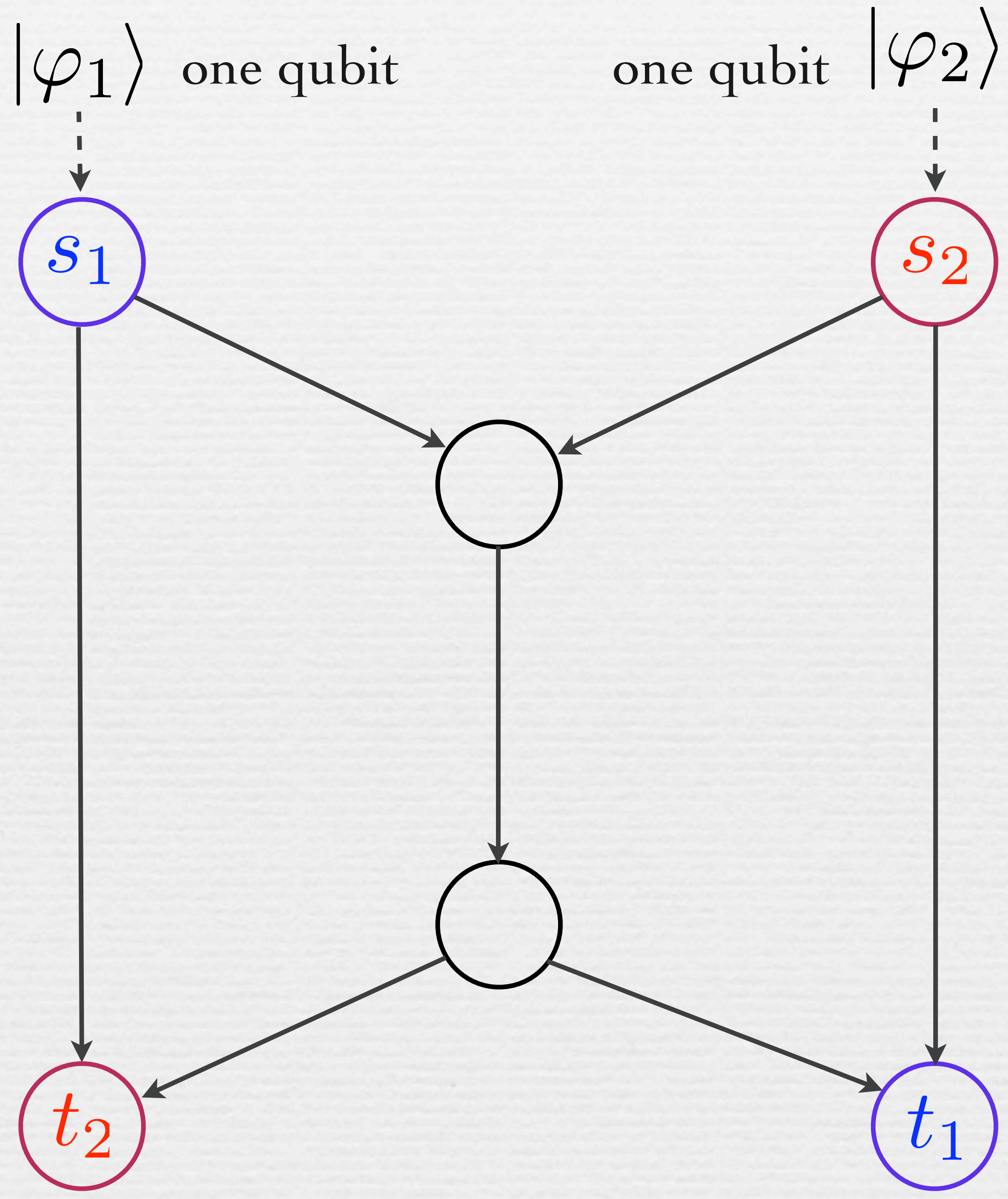
- two targets t_1 and t_2

Goal:

$$\boxed{\begin{array}{l}\text{send } |\varphi_1\rangle \text{ to } t_1 \\ \text{send } |\varphi_2\rangle \text{ to } t_2\end{array}}$$

- each edge has capacity 1 qubit



Quantum Network Coding: Results

On the butterfly graph:

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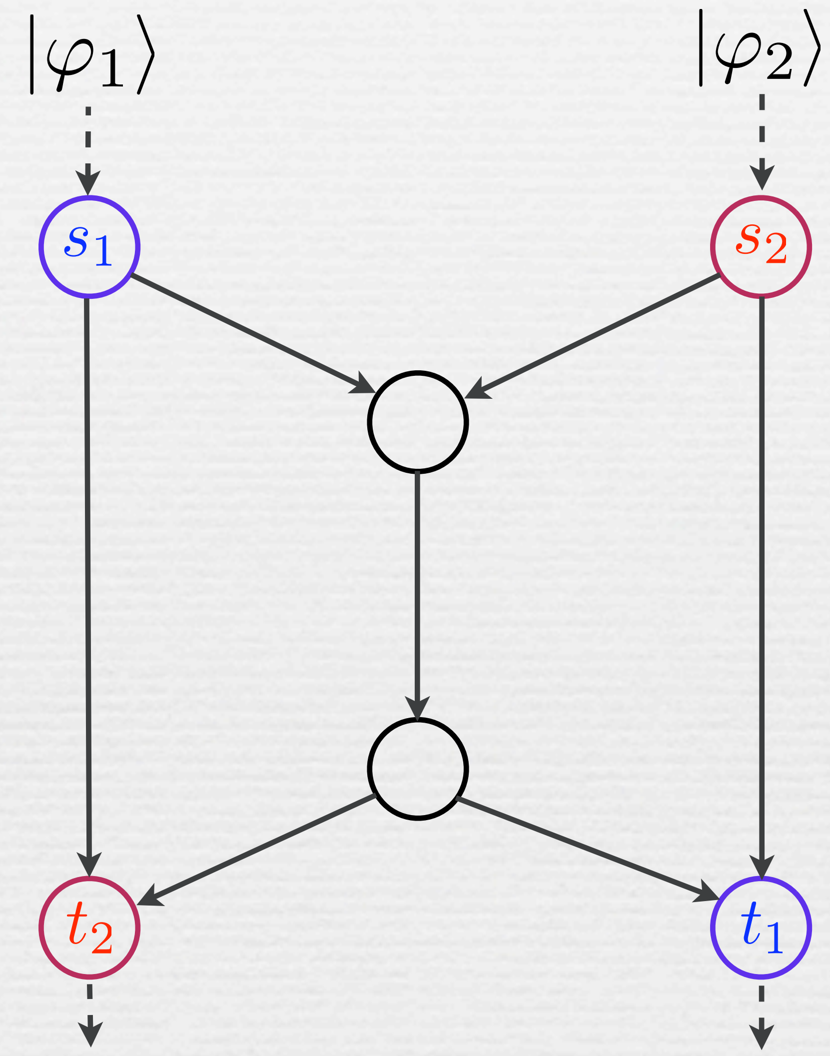

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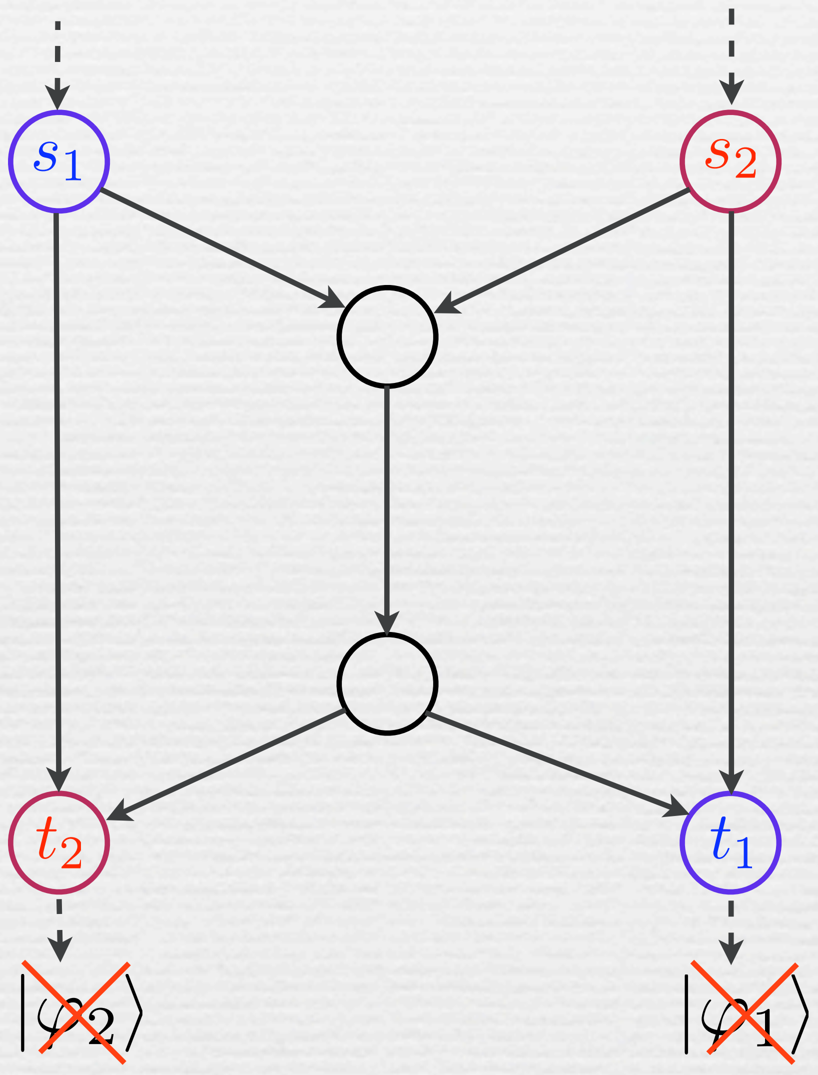

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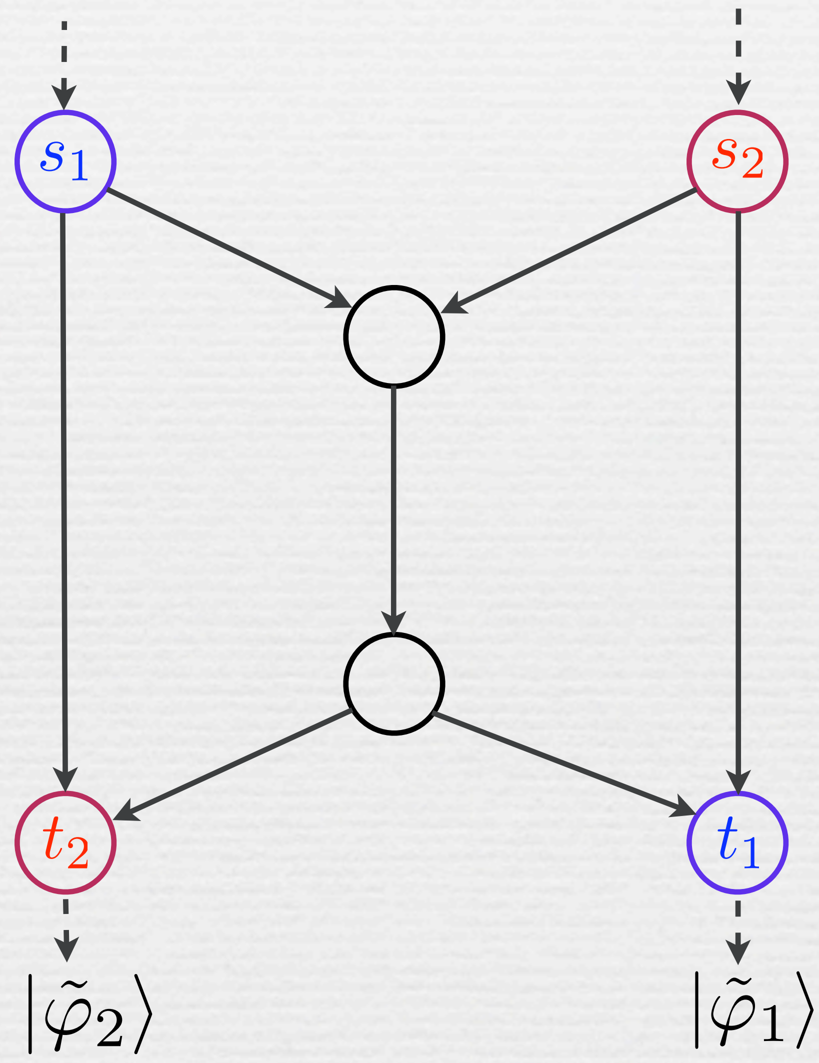

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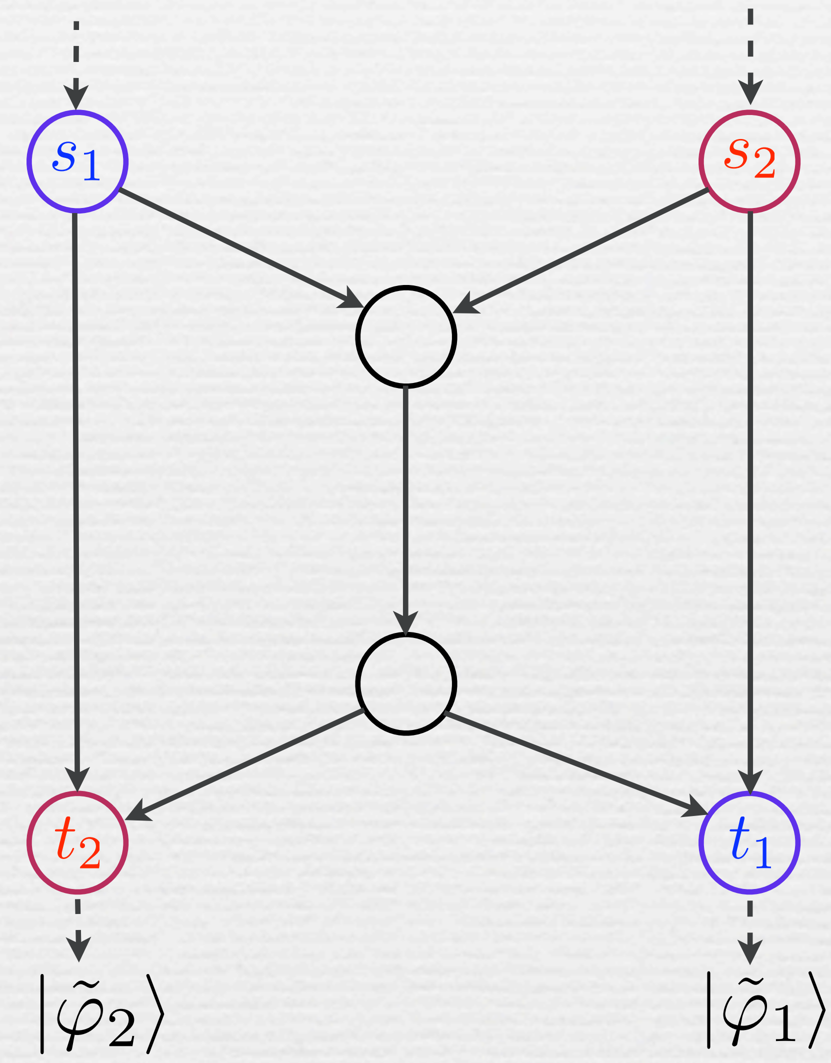

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[Kinji, Murao, Soeda, Turner 2010] $\longleftarrow$ Monday's poster session

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without additional resource

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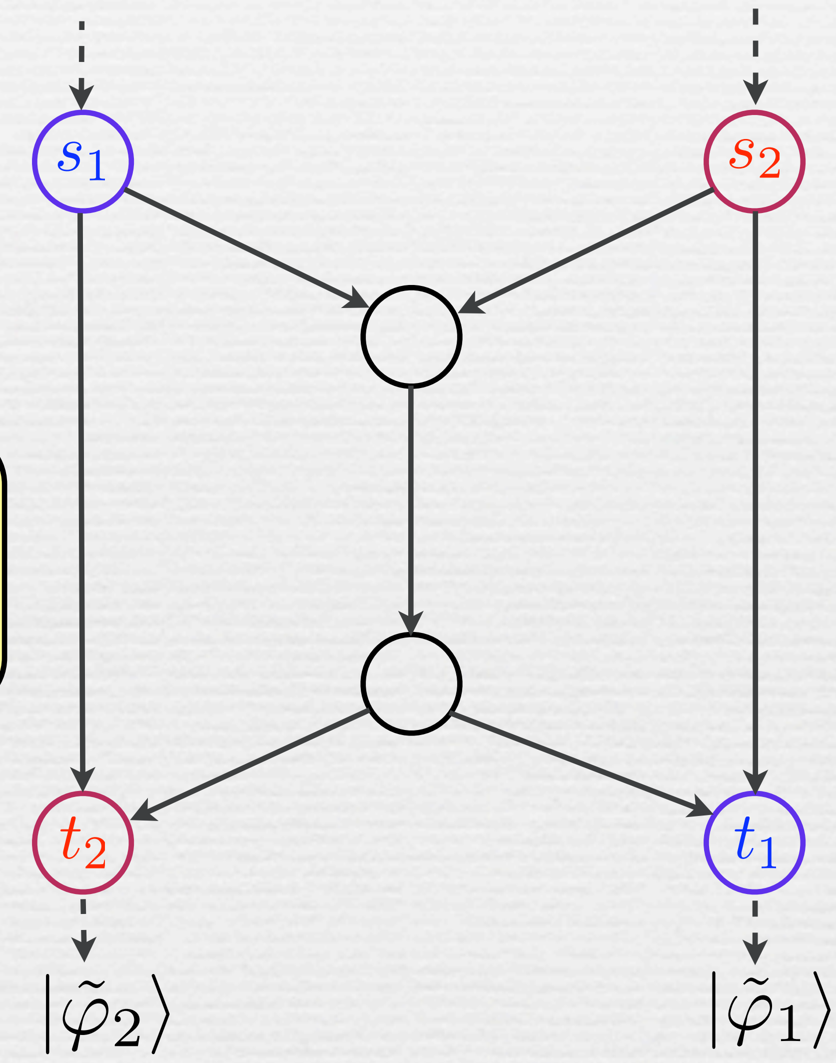

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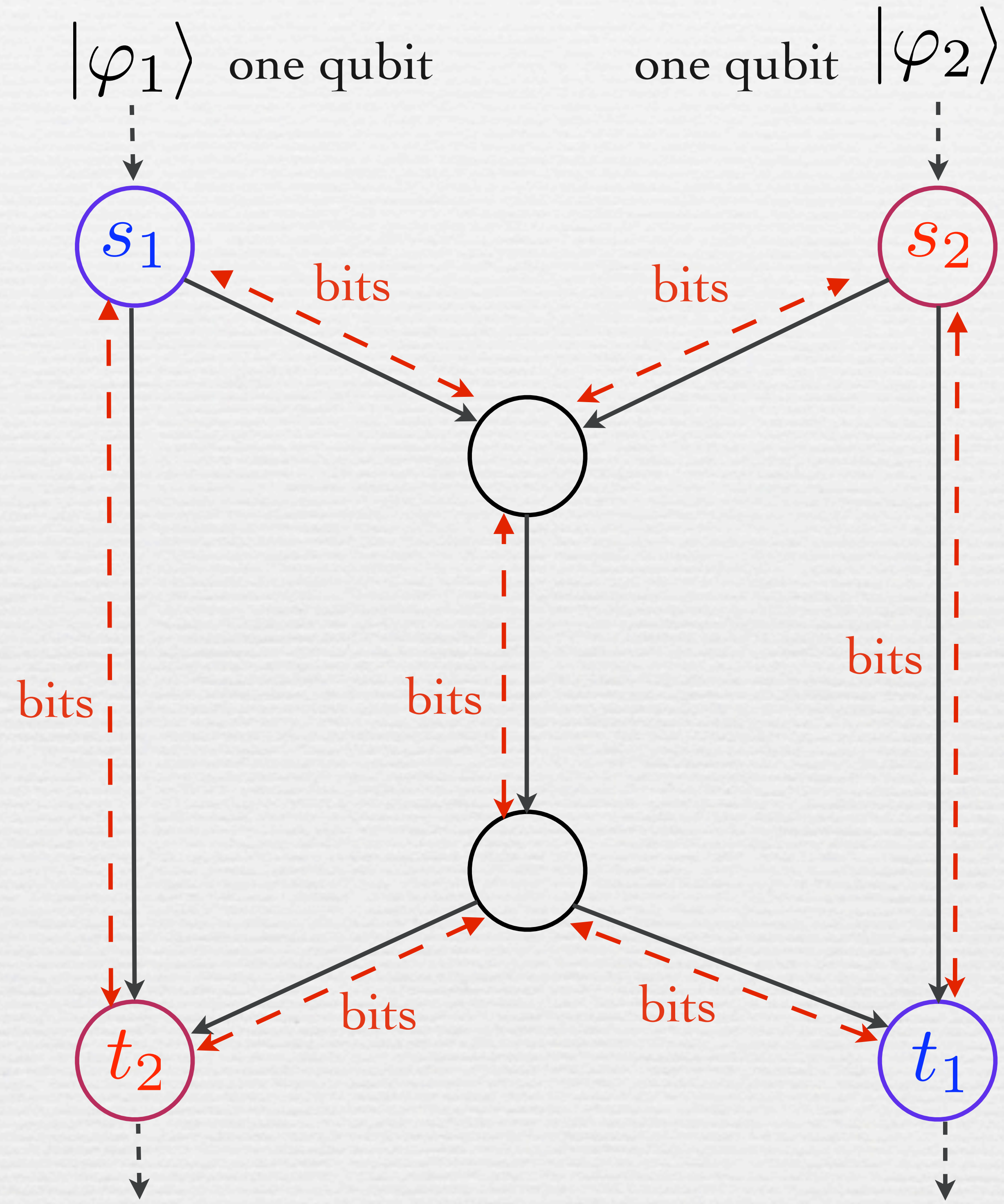
Statement of our Results

Our Setting

- we allow free classical communication between any pair of adjacent nodes



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preliminary result

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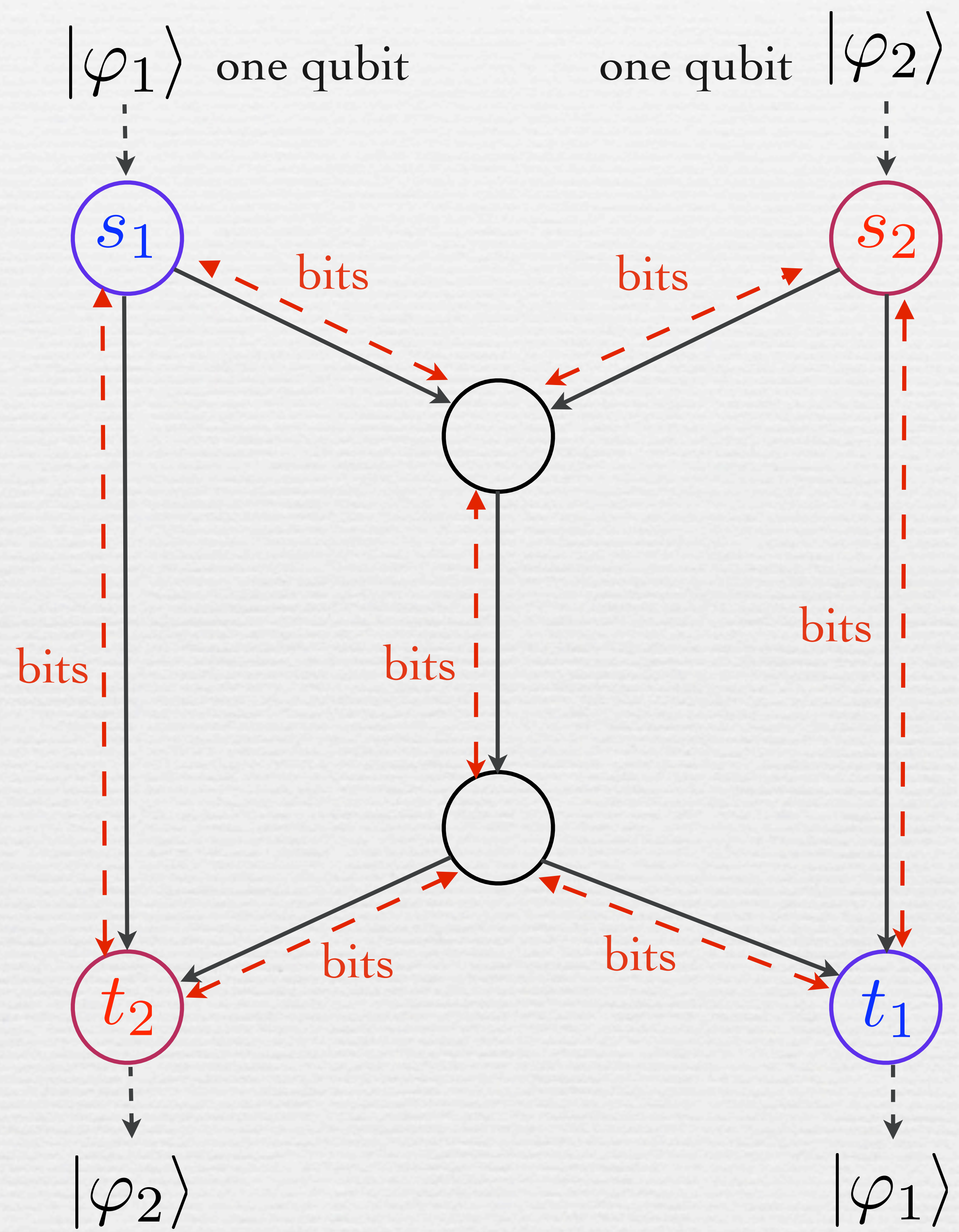

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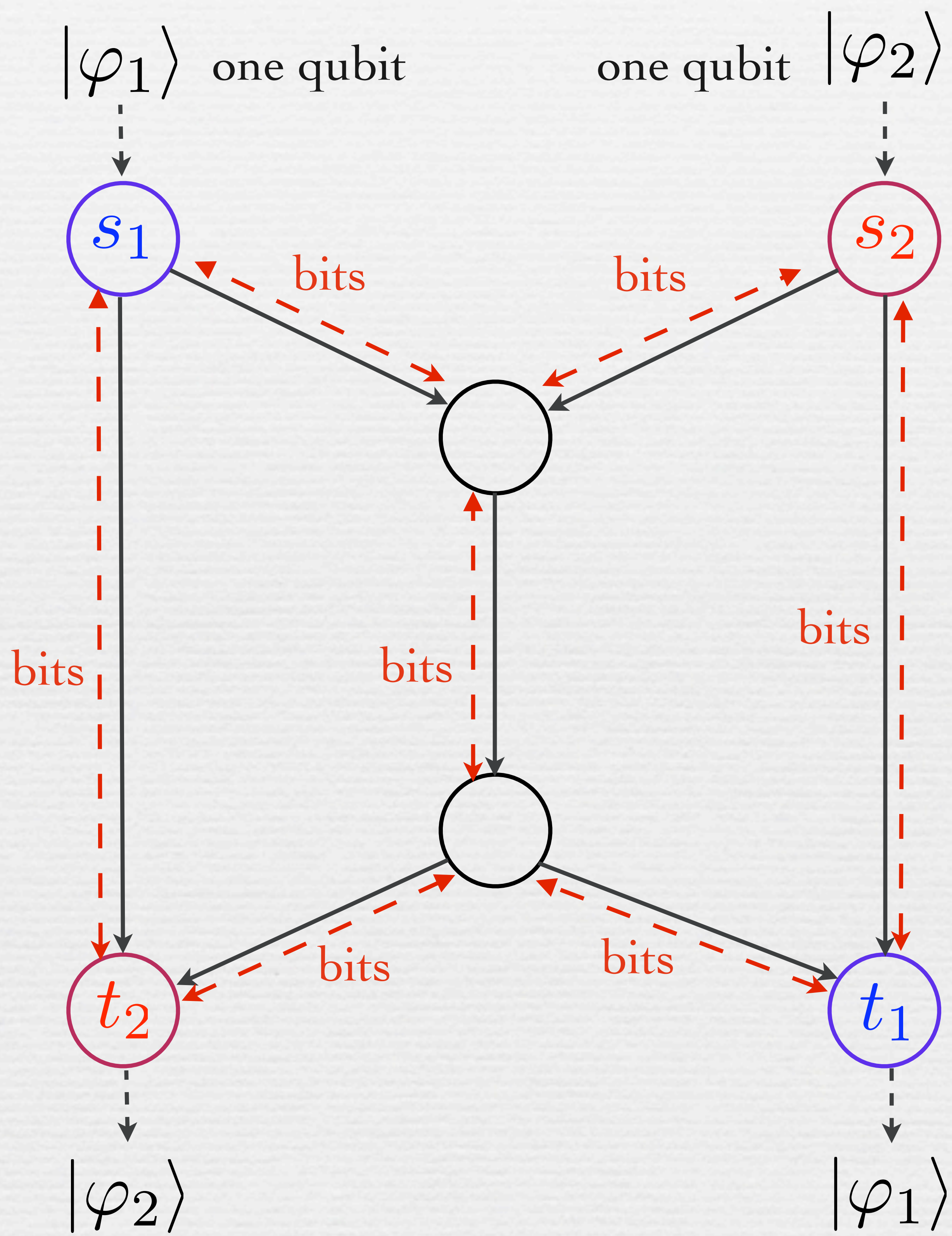

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quantum perfect network coding is possible on the butterfly graph

general result

this is true for any graph

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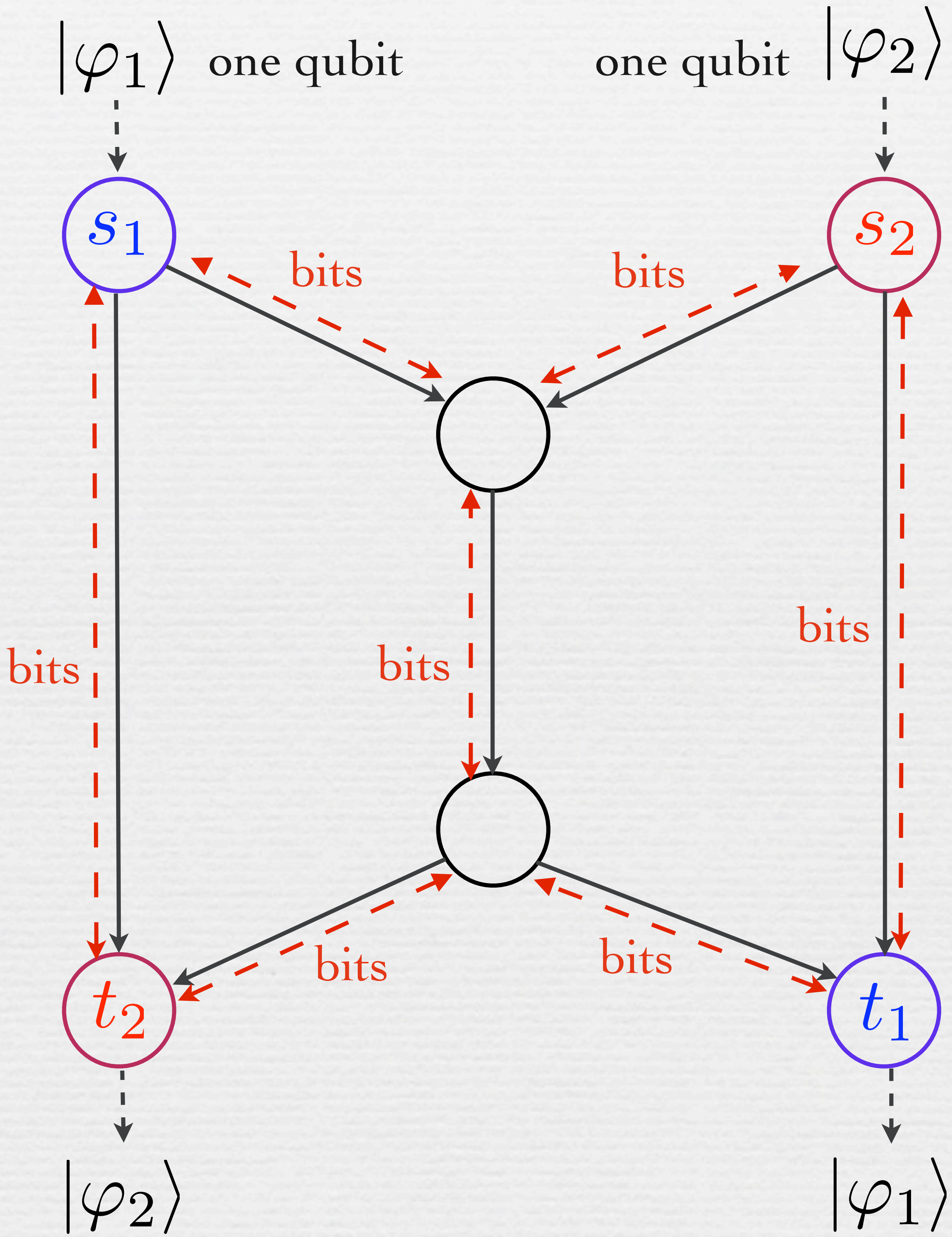
general result

this is true for any graph

- reasonable hypothesis: classical communication is much cheaper than quantum communication

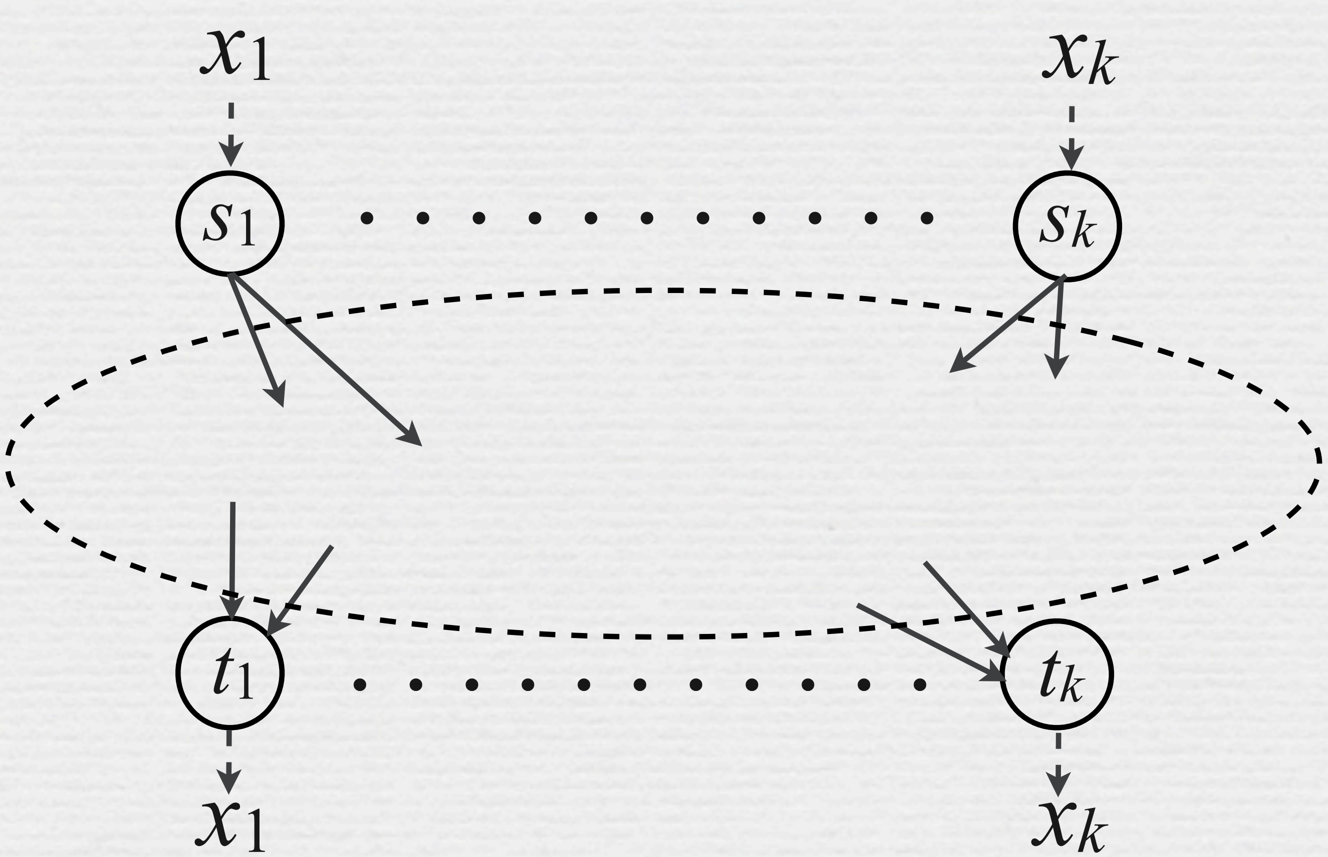


The Classical k-pair Problem

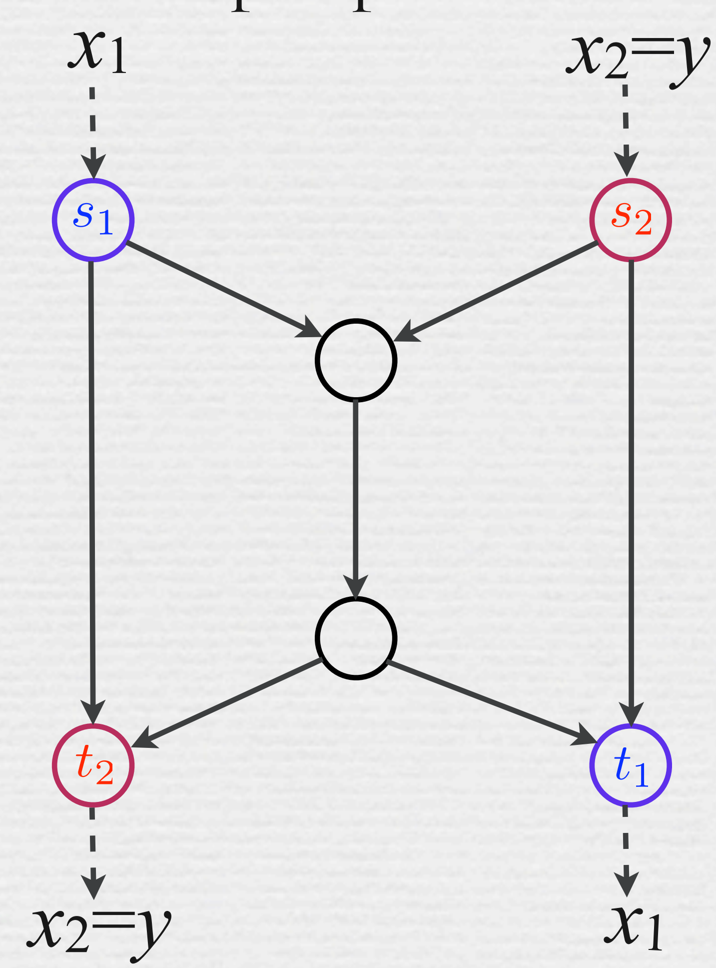
given: • a directed (acyclic) graph
• k source nodes $s_1, \ldots s_k$
• k target nodes $t_1, \ldots t_k$

goal: one bit x_i has to be sent from s_i to t_i

each edge has capacity 1



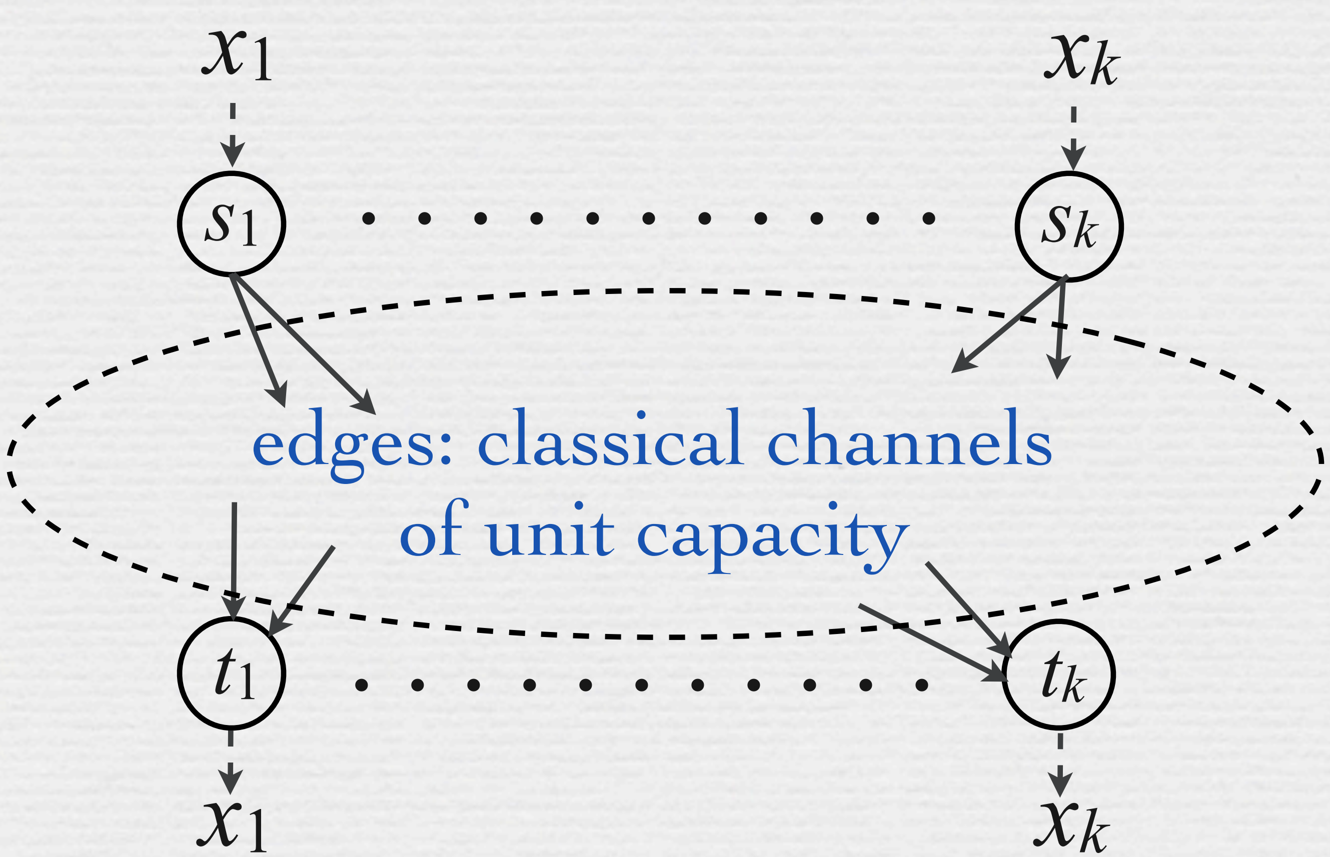
butterfly: instance of the 2-pair problem



Main Result

Suppose that a given instance of the classical k-pair problem has a solution.

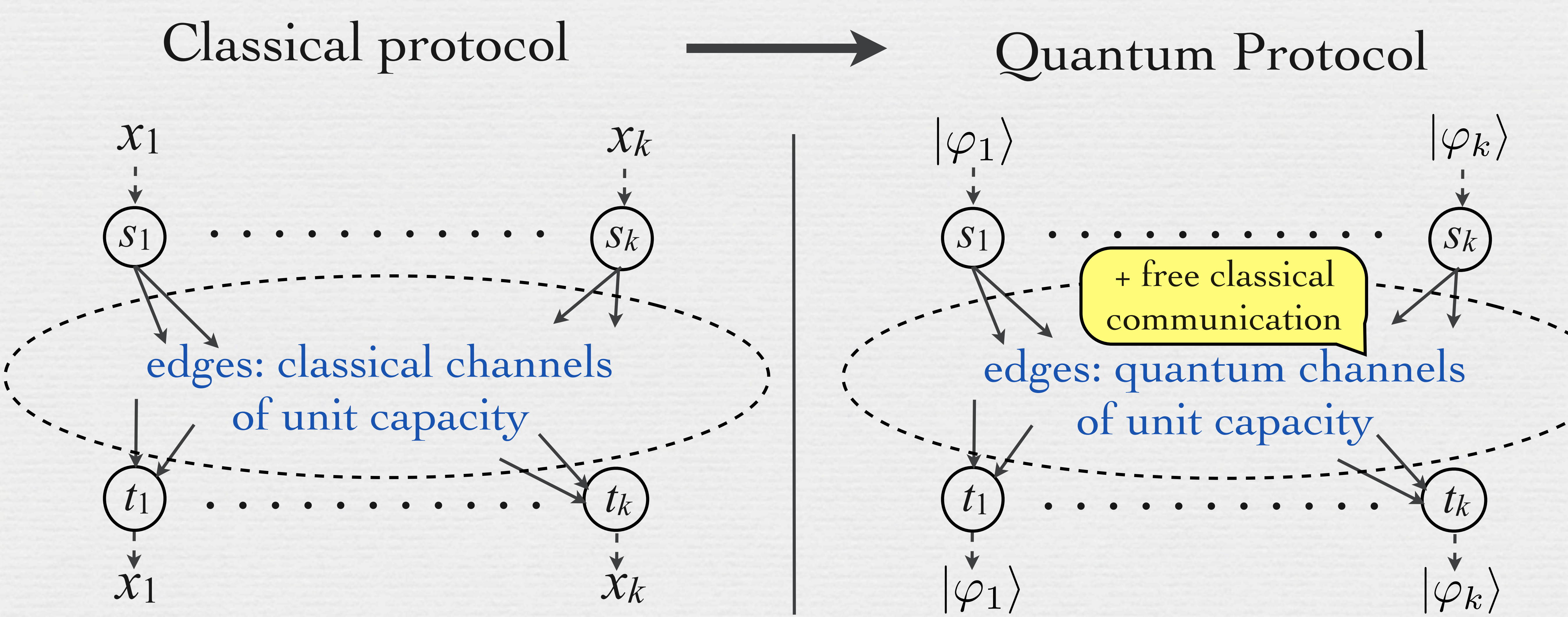
Classical protocol



Main Result

Main Theorem

Suppose that a given instance of the classical k-pair problem has a solution.
Then the associated quantum instance has a perfect solution if free classical communication is allowed.



Relation with our Previous Work

This result improves and generalizes our previous work [KLNR 2009]

arXiv:0908.1457 and ICALP'09

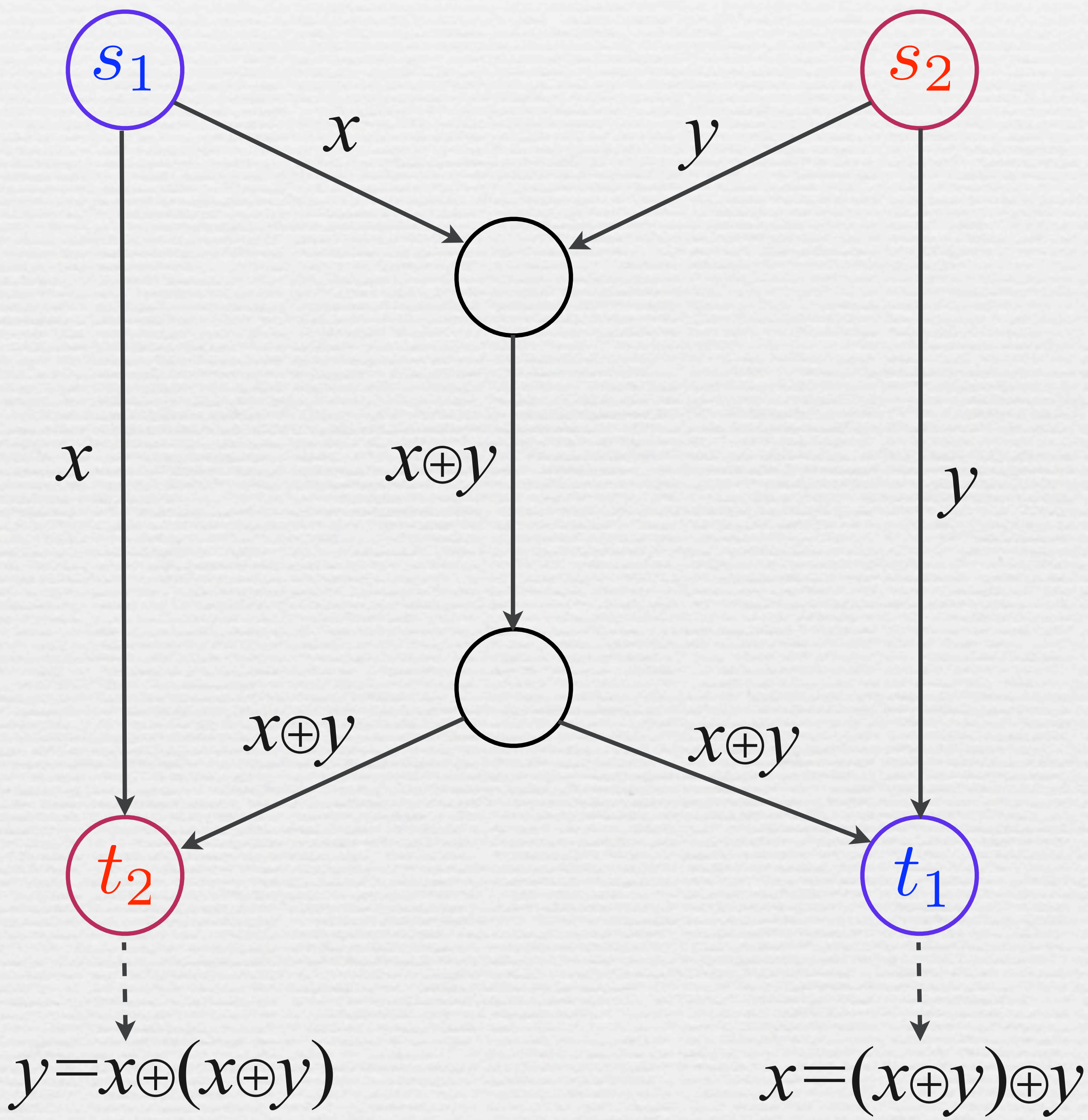
	[KLNR'09]	This talk
number of bits of free classical communication sent per edge	polynomial	≤ 2
condition on the classical protocol	linear protocol	none

Note: there exist solvable classical *k*-pair problems for which no linear protocol exists [Dougherty, Freiling and Zeger 2005] [Riis 2003]

Illustration on the Butterfly Graph

Quantum Protocol

classical coding scheme



quantum simulation

Three steps:

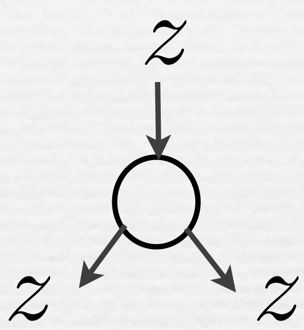
I. node-by-node simulation

II. removal of internal registers

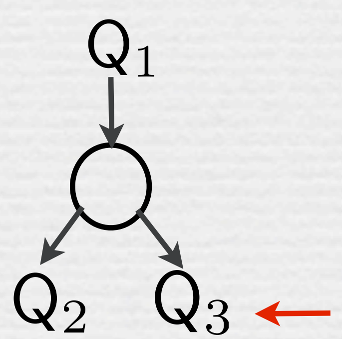
III. removal of initial registers

I. node-by-node simulation

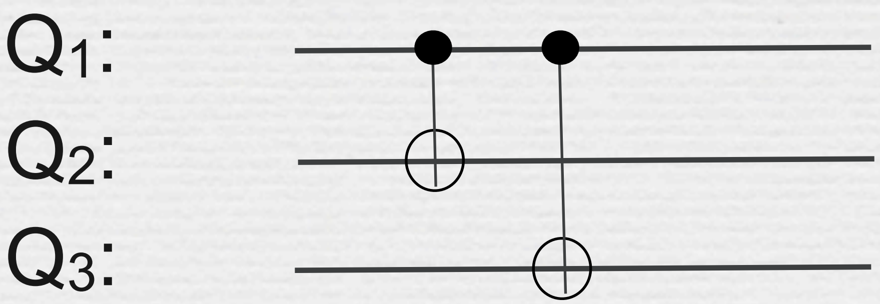
classical copy node:



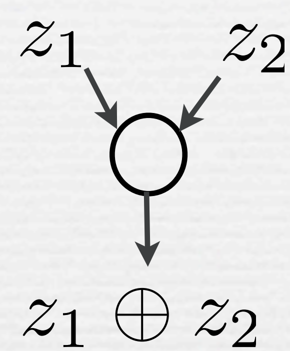
quantum simulation:



two new registers
initialized to $|0\rangle$



classical parity node:



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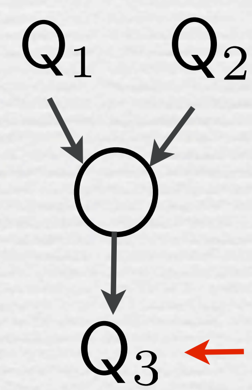
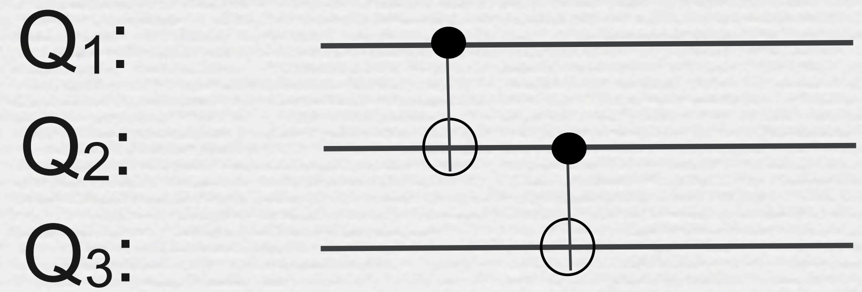


new register initialized to $|0\rangle$



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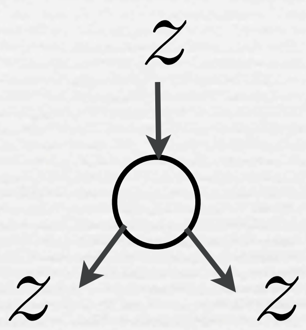

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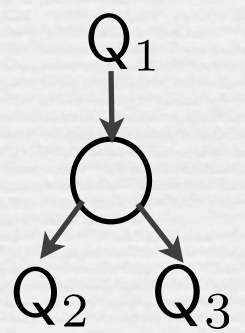
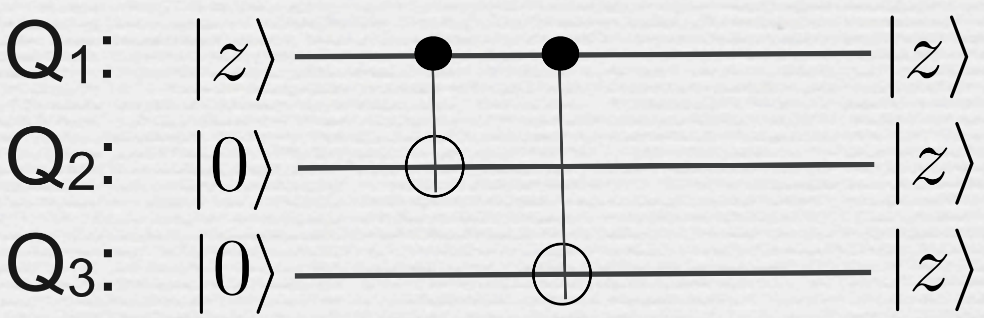


two new registers initialized to $|0\rangle$

$z = 0, 1$



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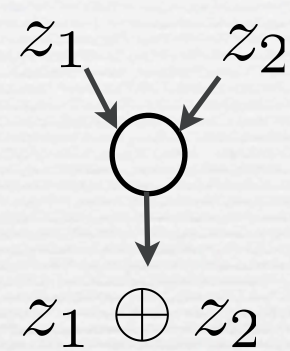

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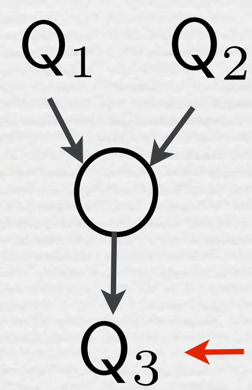

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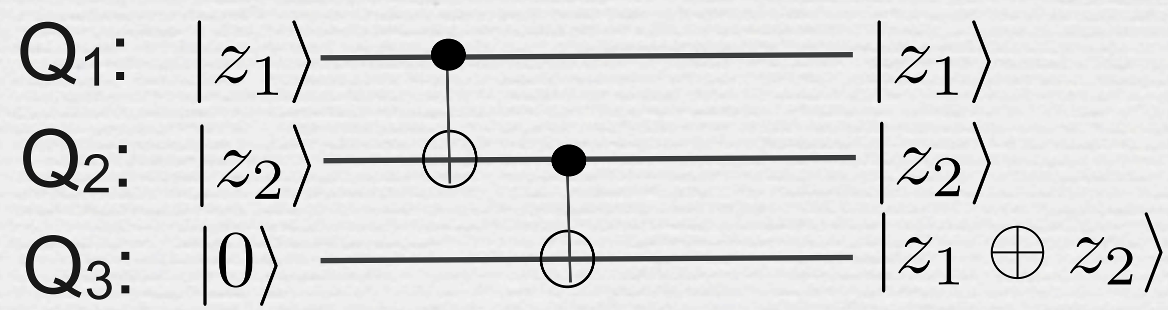
$z_1, z_2 = 0, 1$



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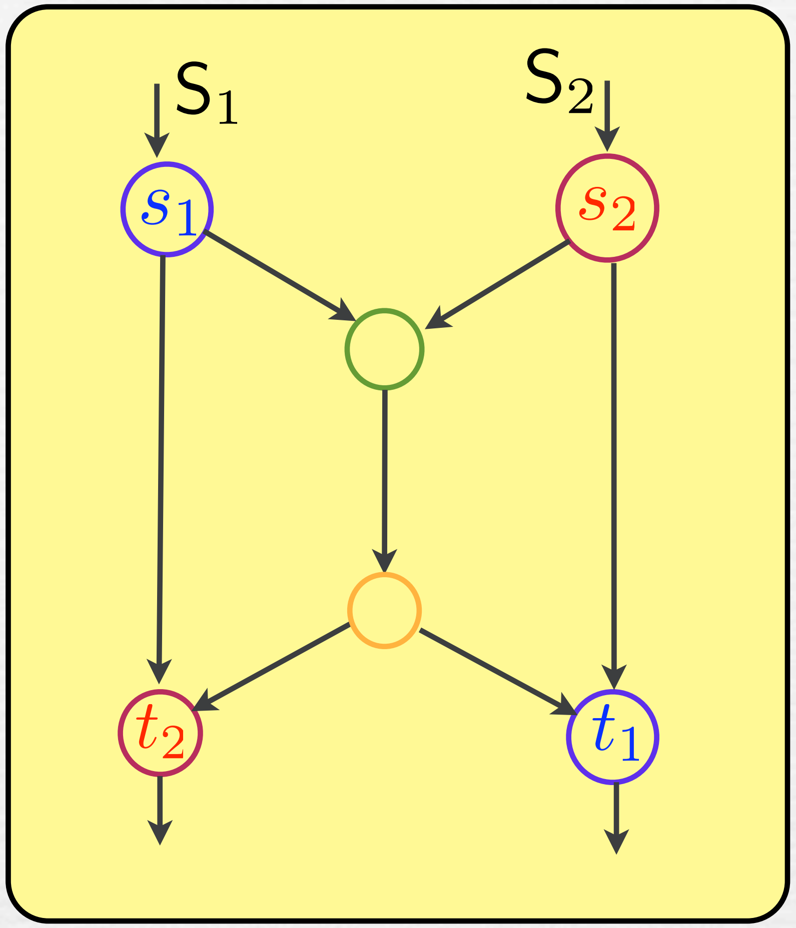

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basis state (for now)

$x, y \in \{0, 1\}$

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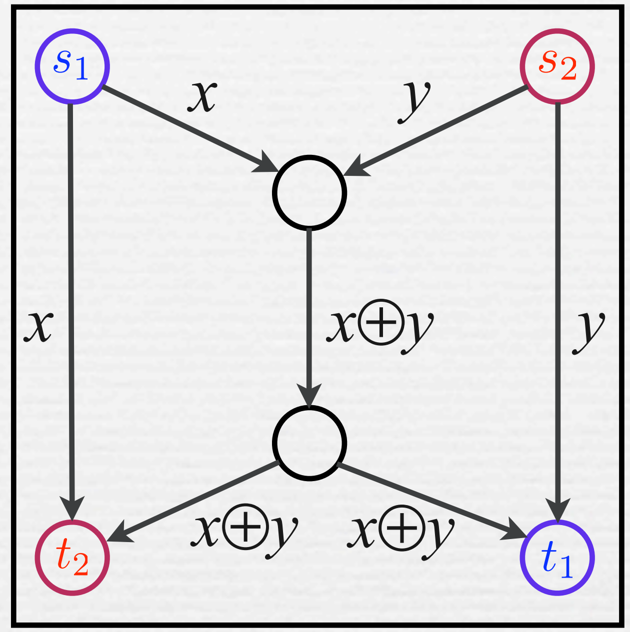

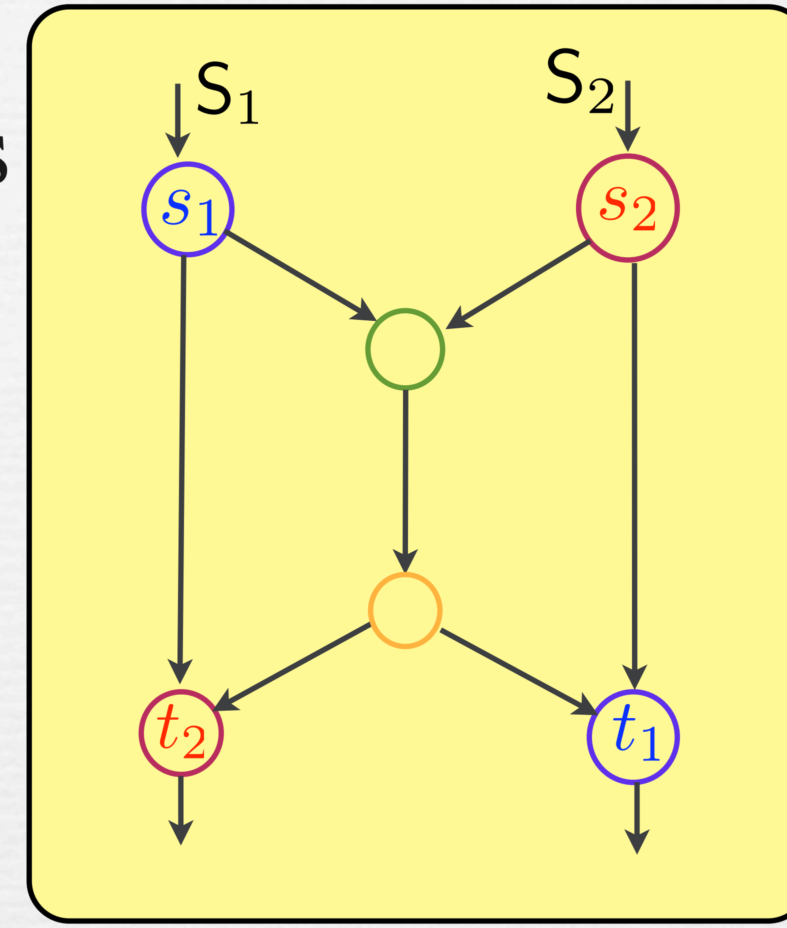

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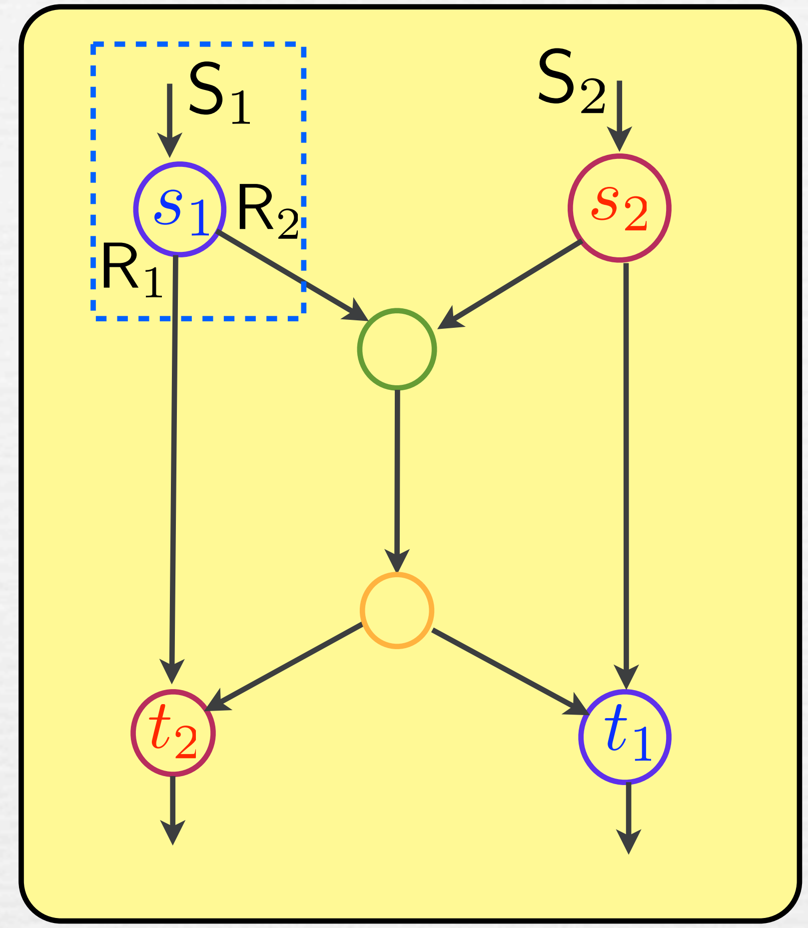

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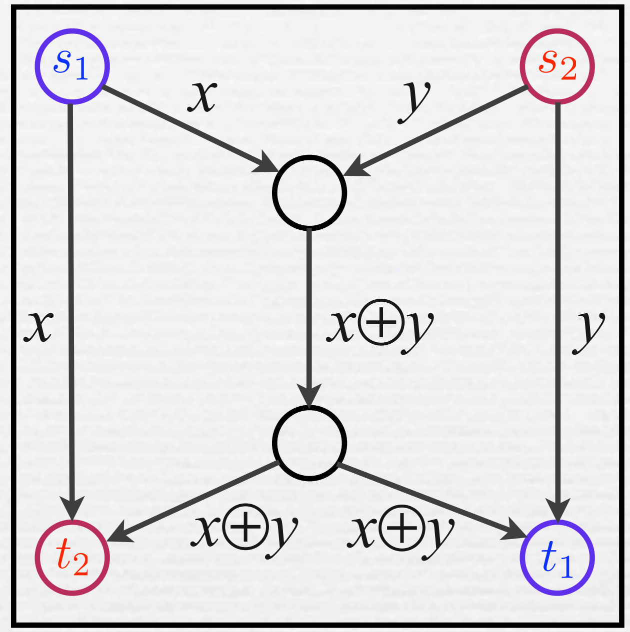

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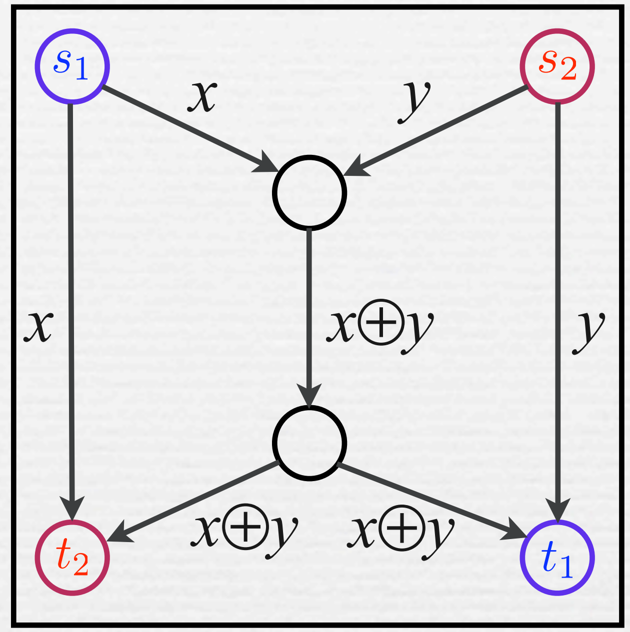

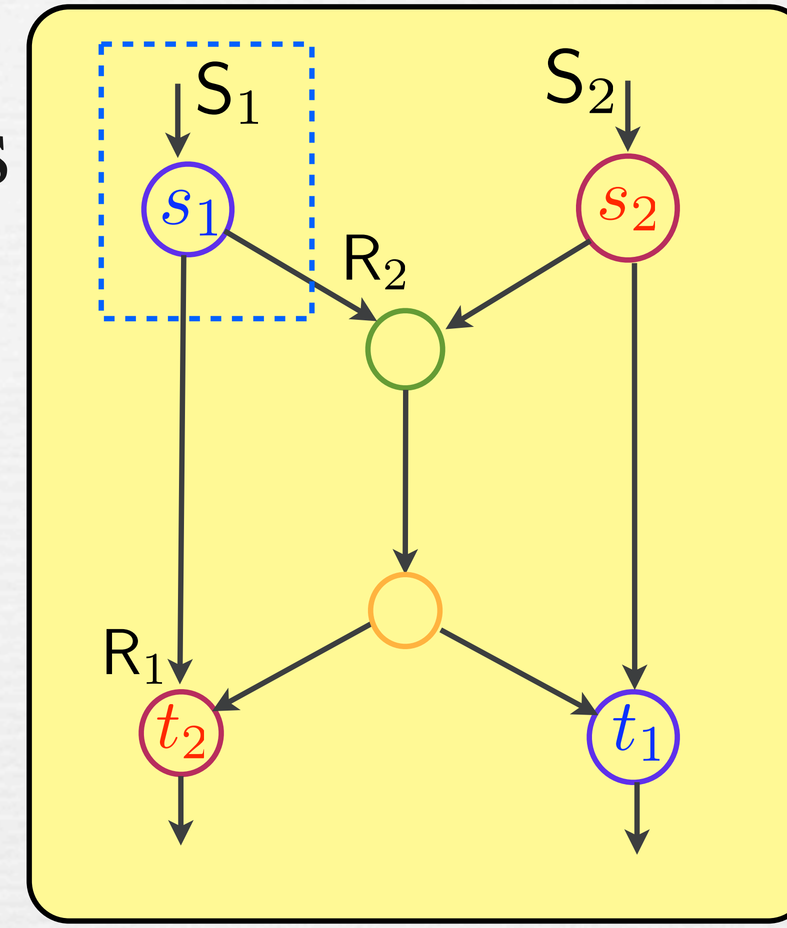

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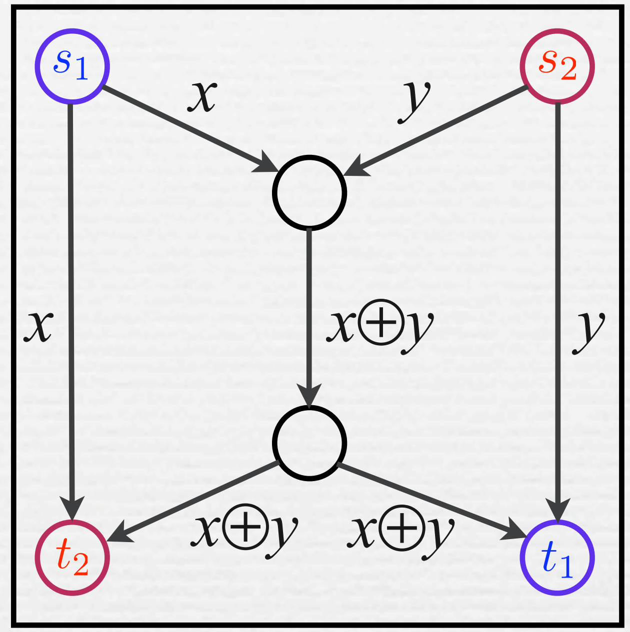

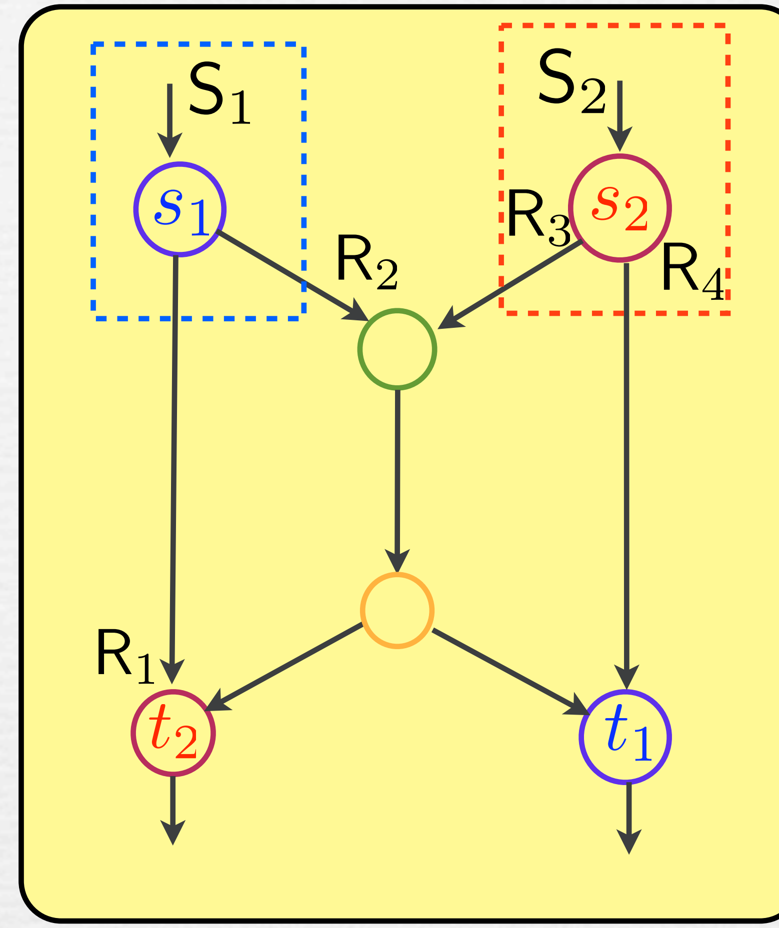

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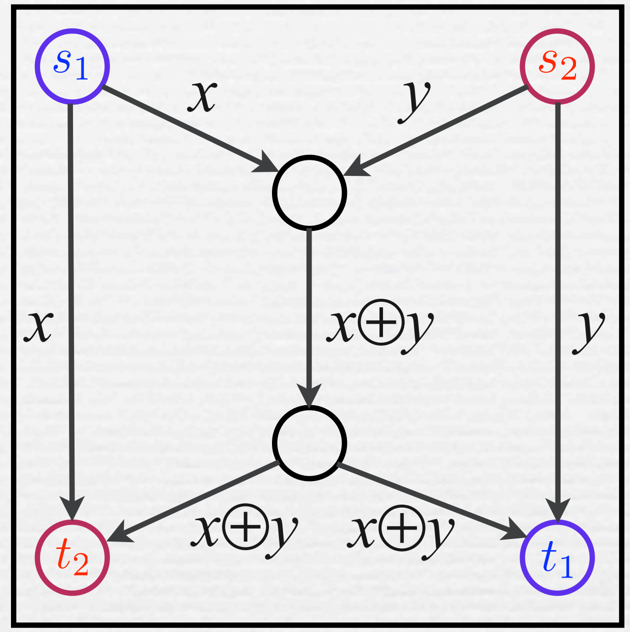

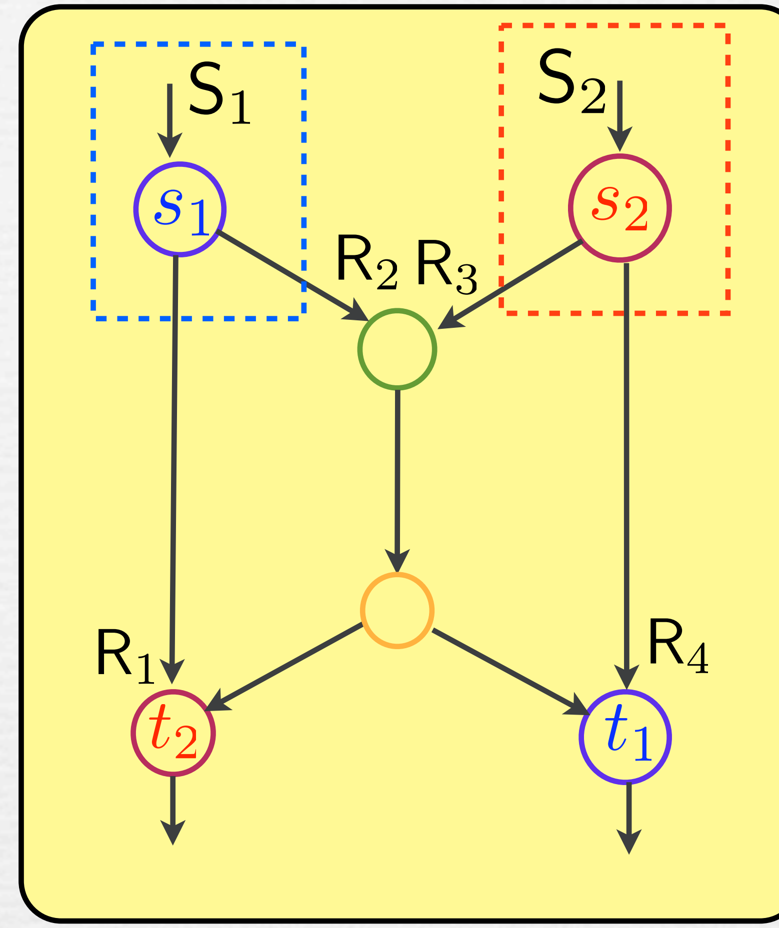

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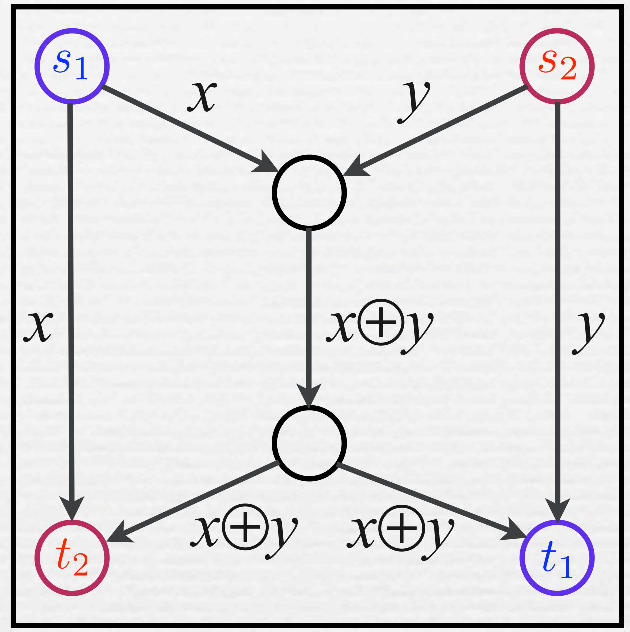

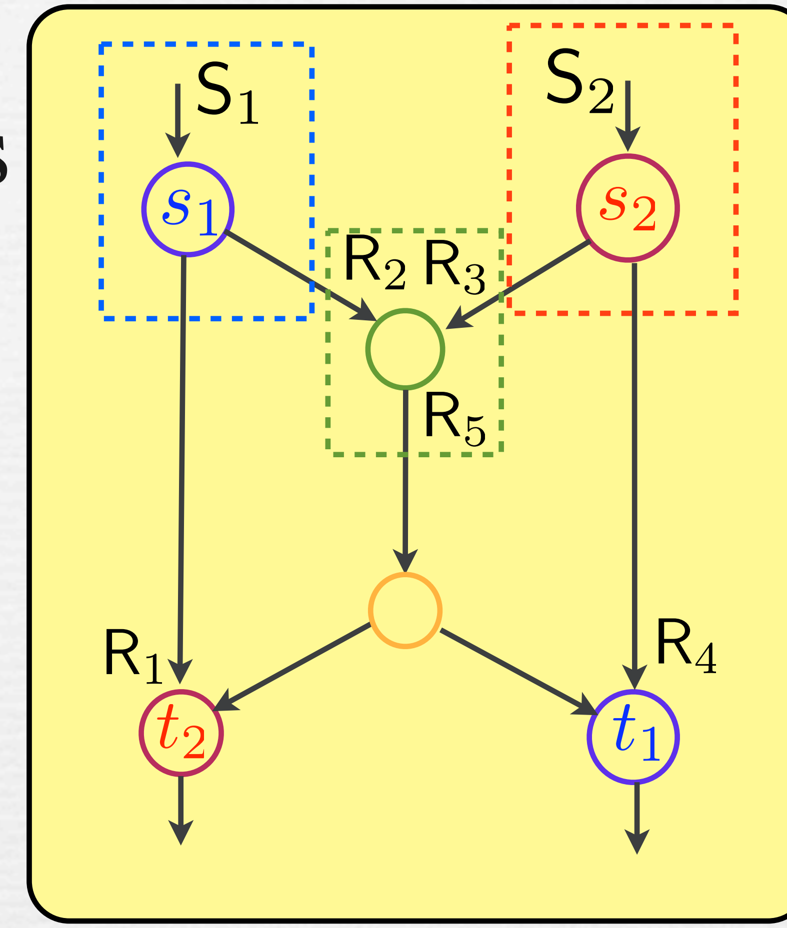

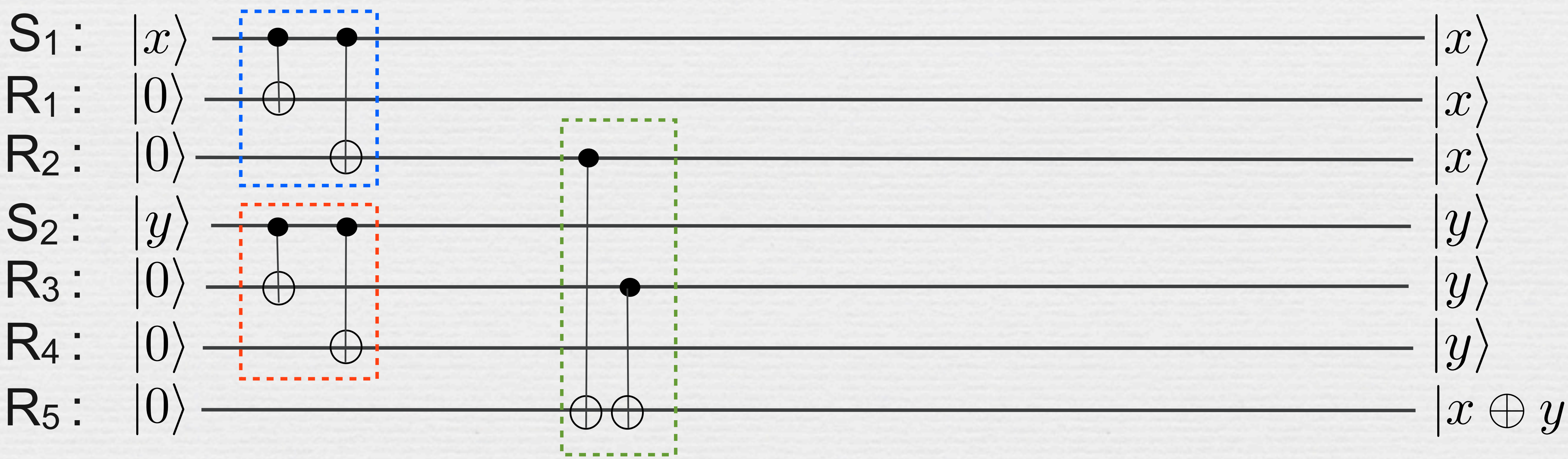

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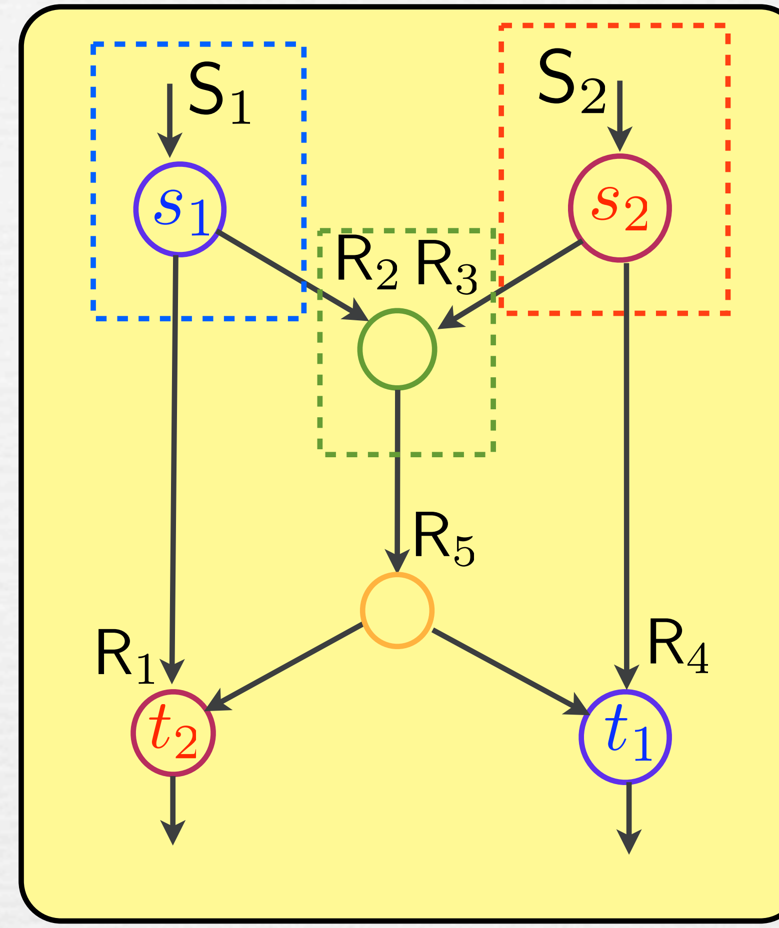

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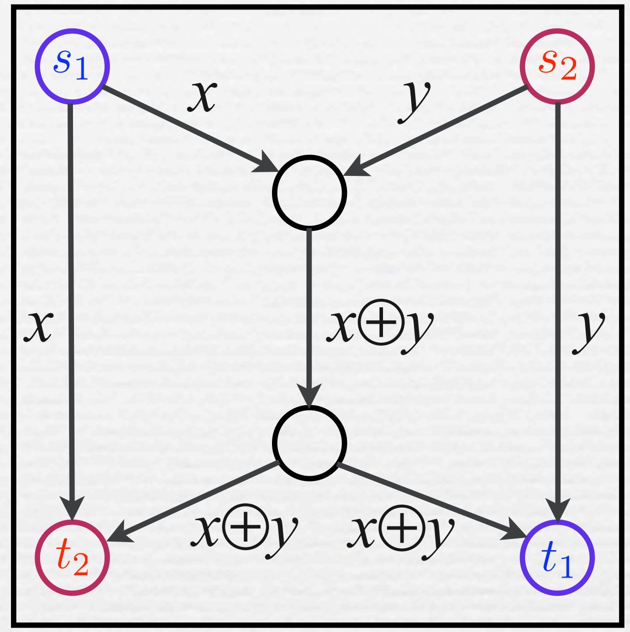

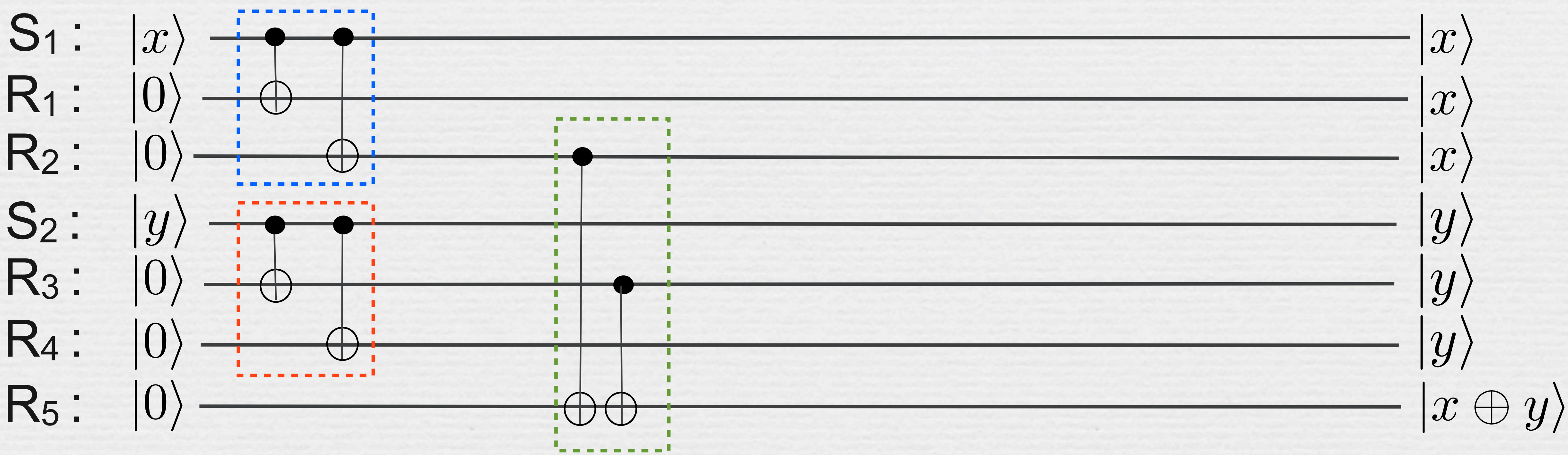

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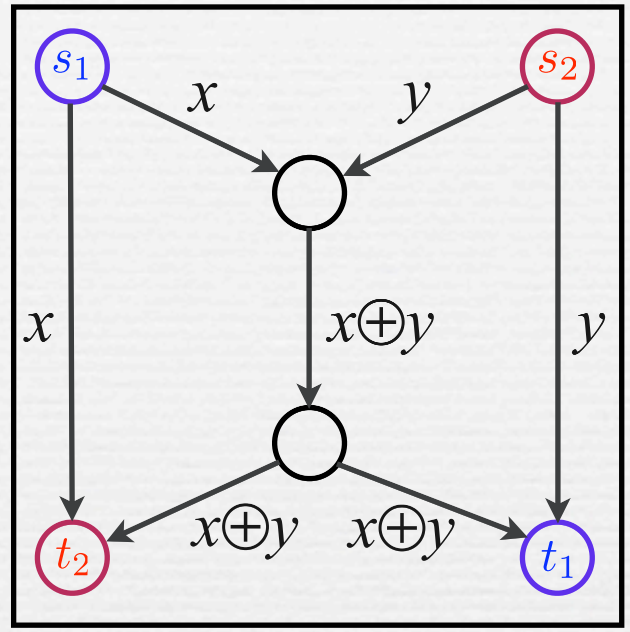

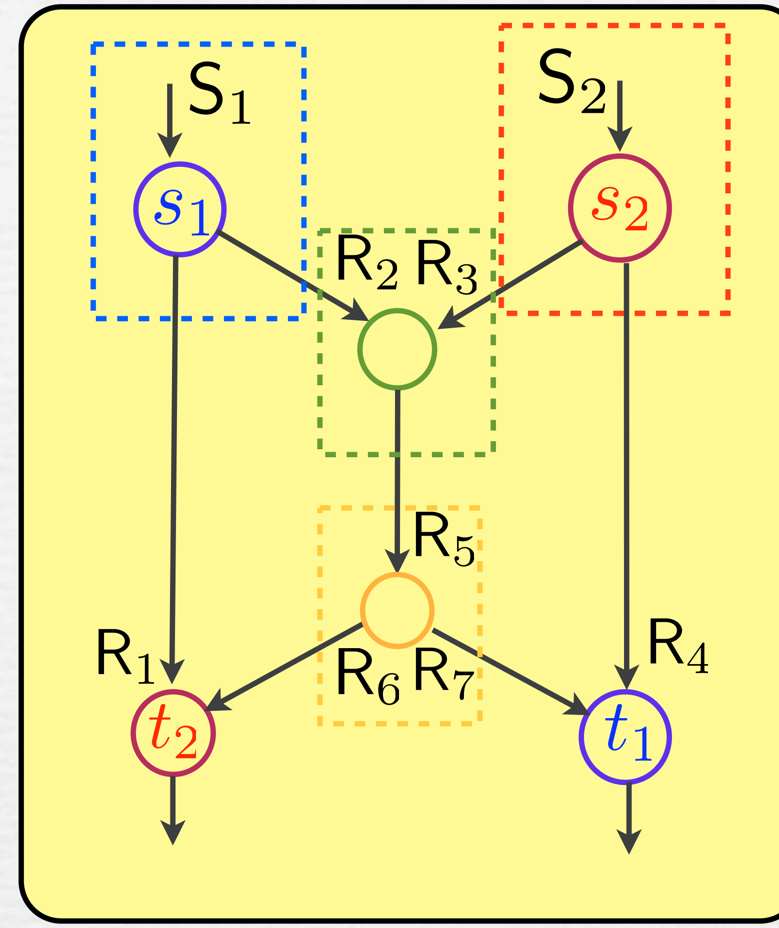

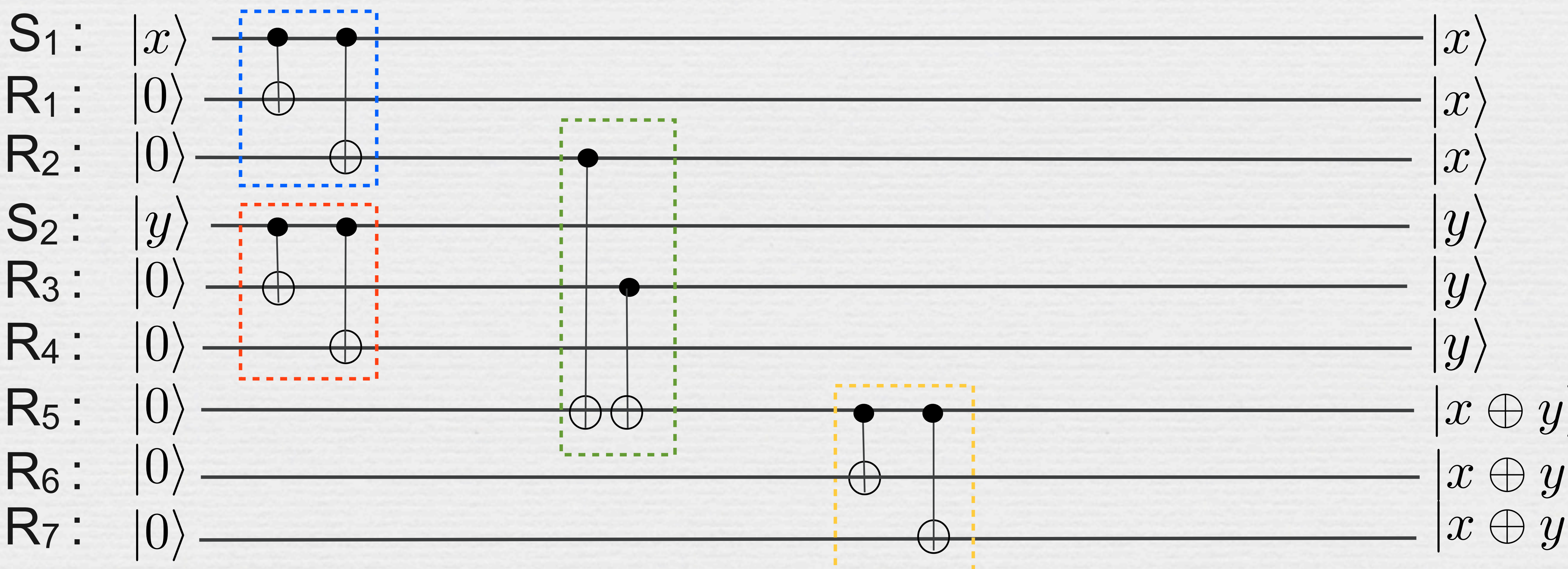

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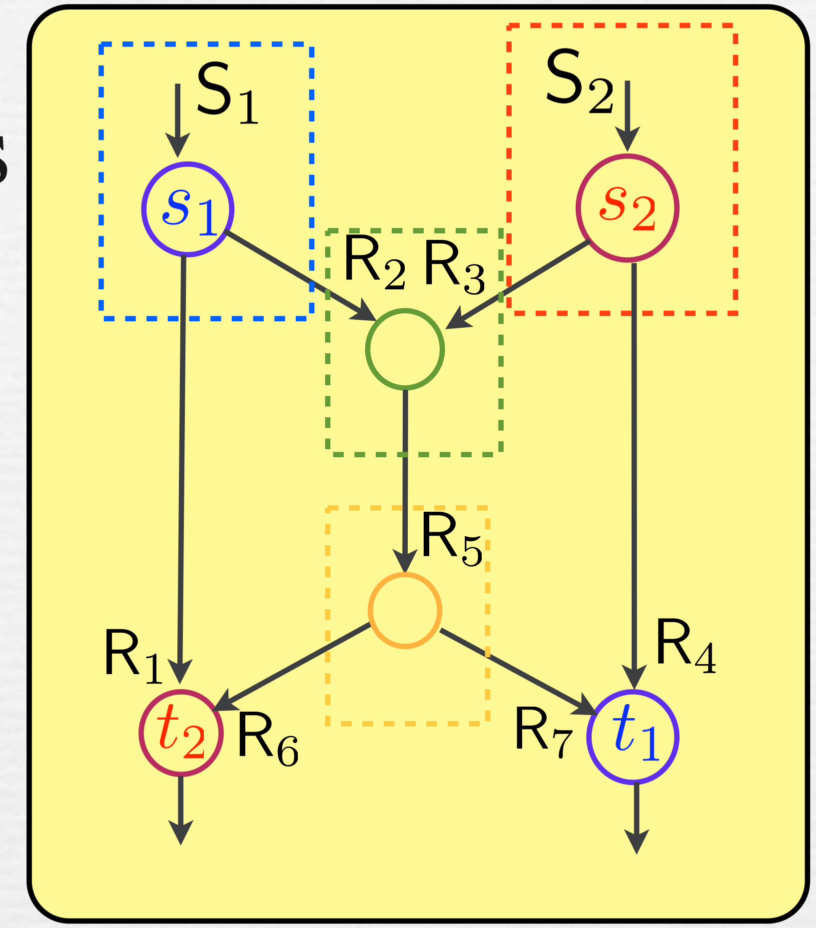

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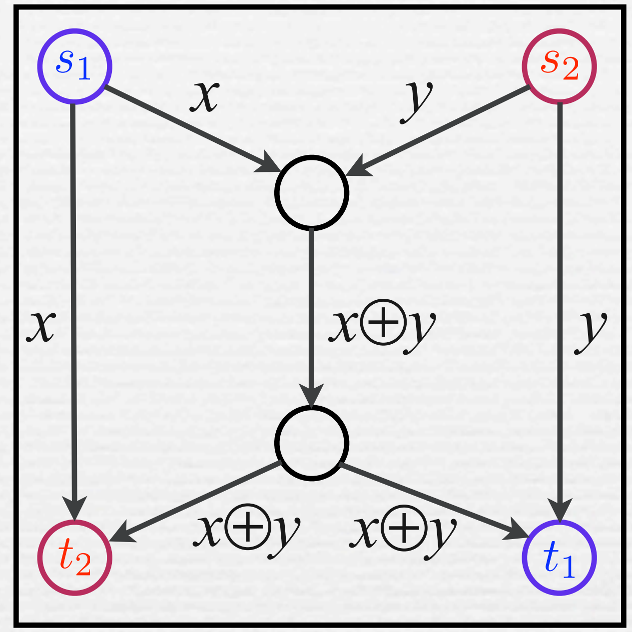

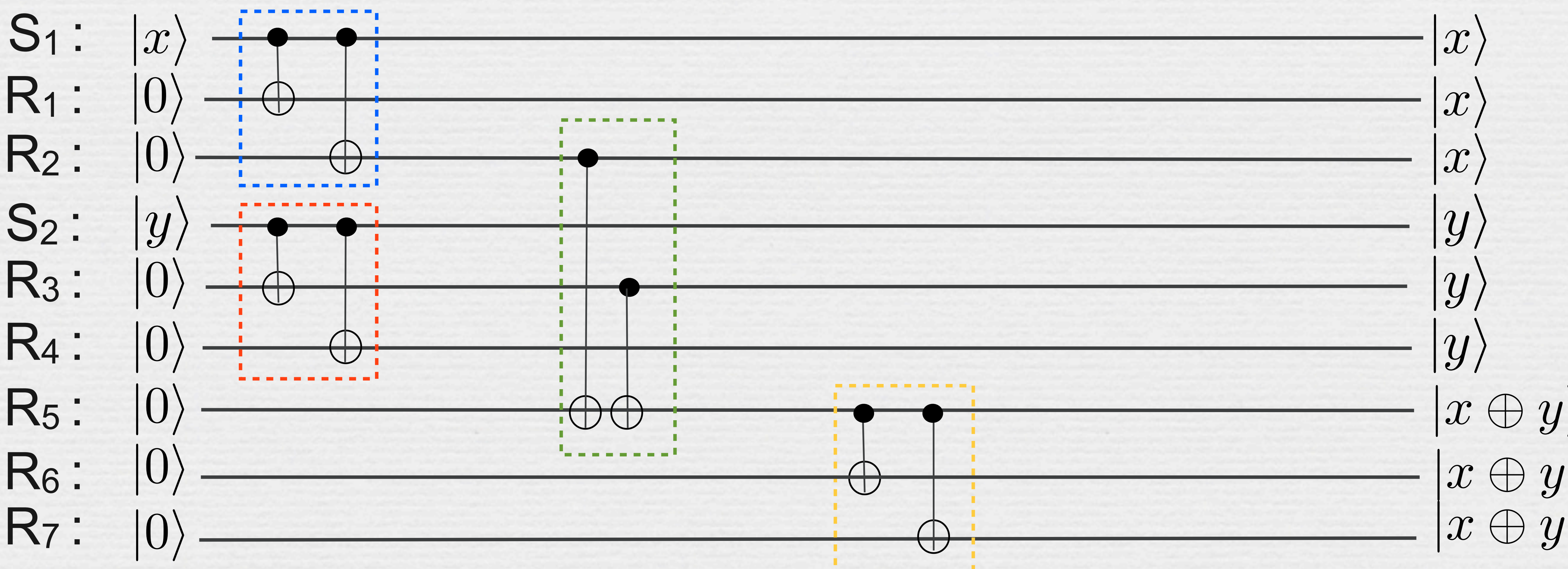

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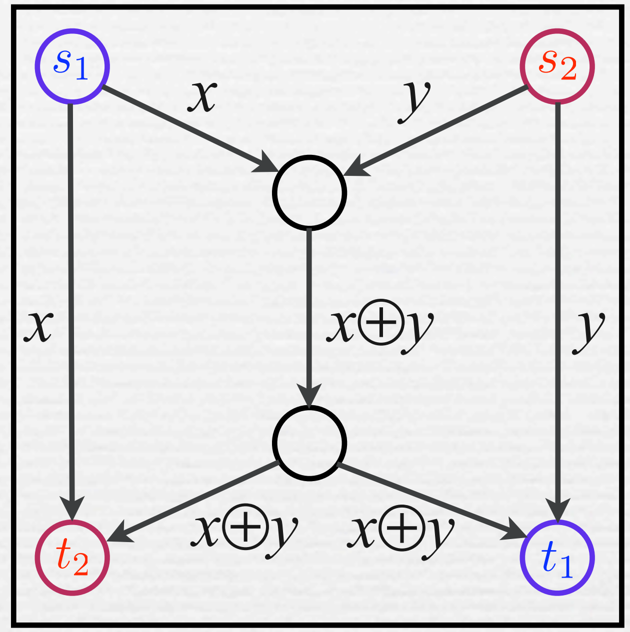
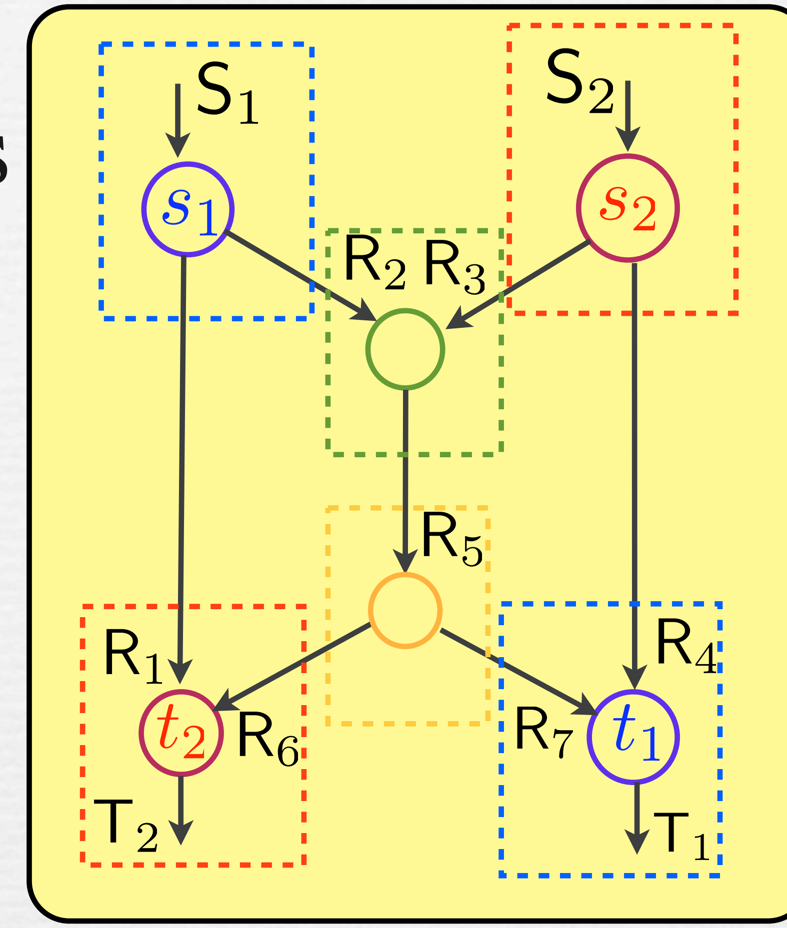
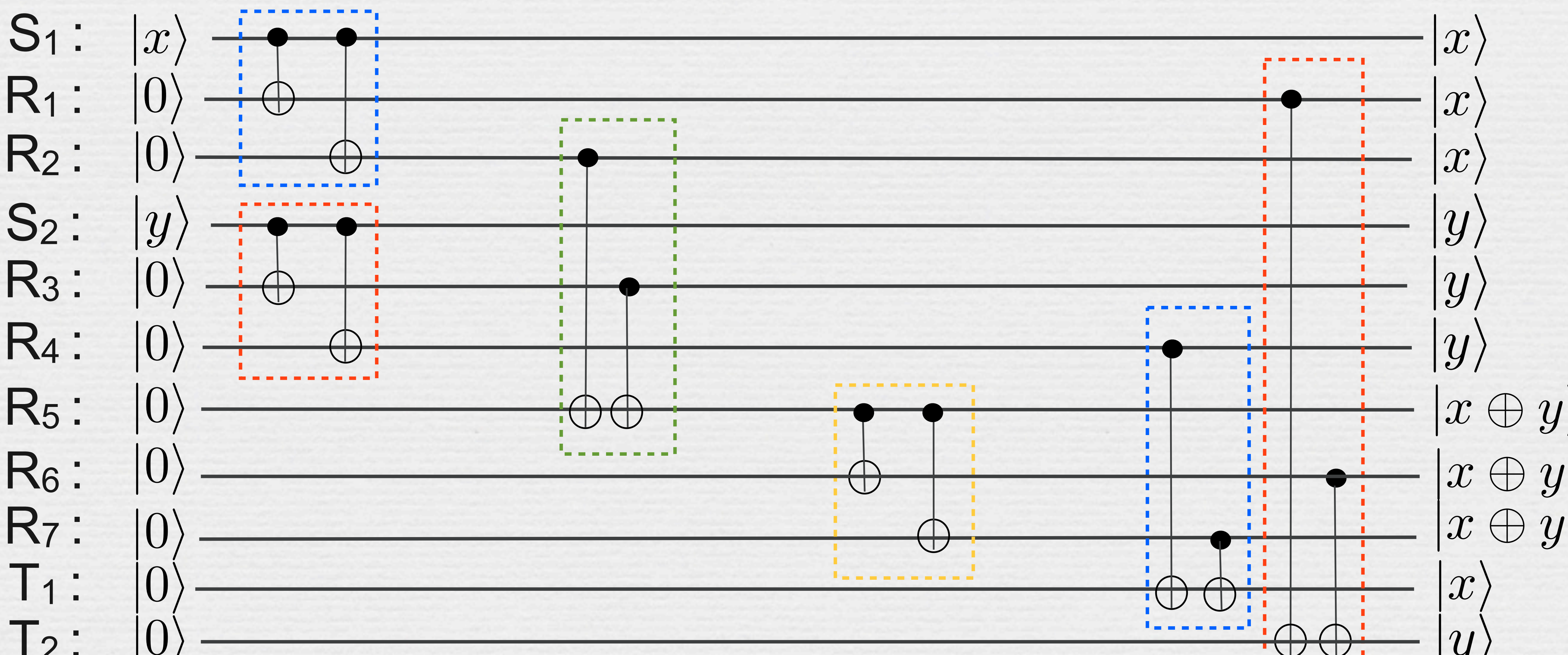
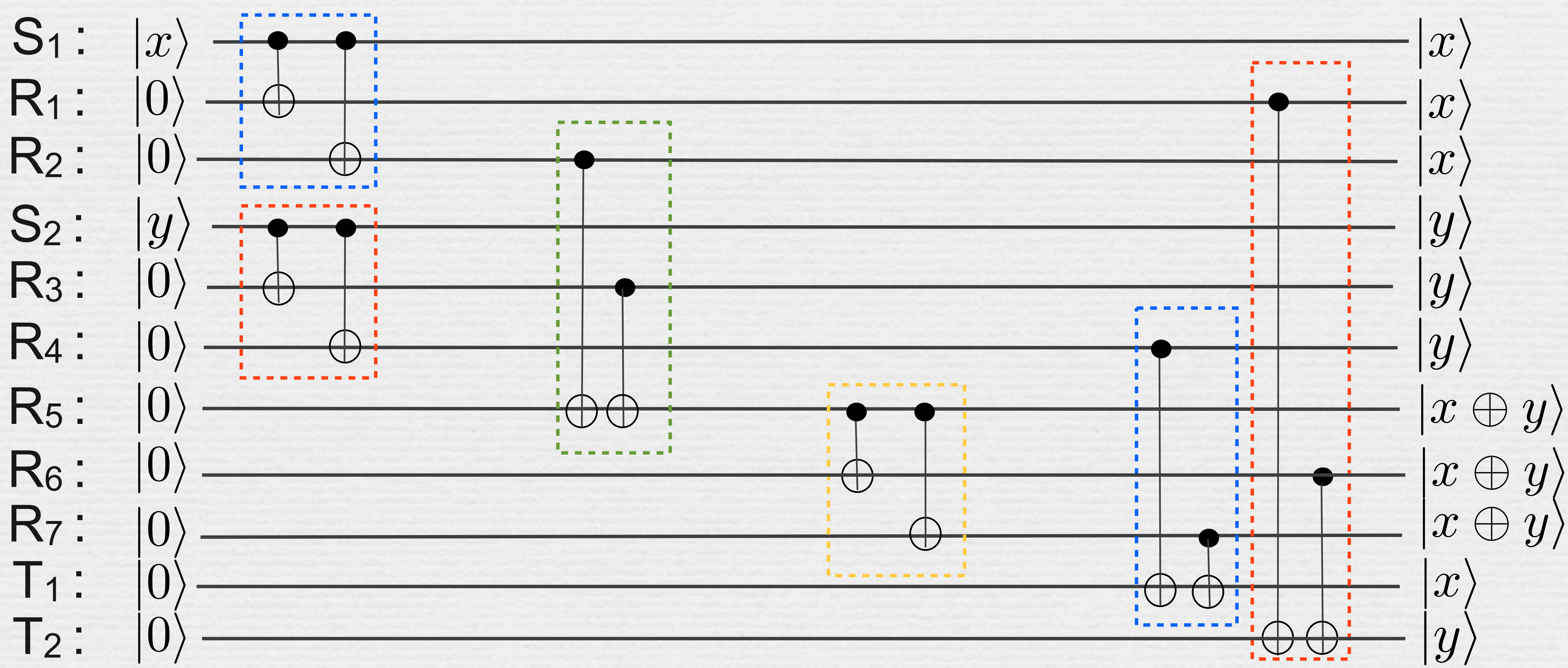






initial state: $$\sum_{x,y\in\{0,1\}} \alpha_{xy}|x\rangle_{\mathsf{S}_1}|y\rangle_{\mathsf{S}_2}$$

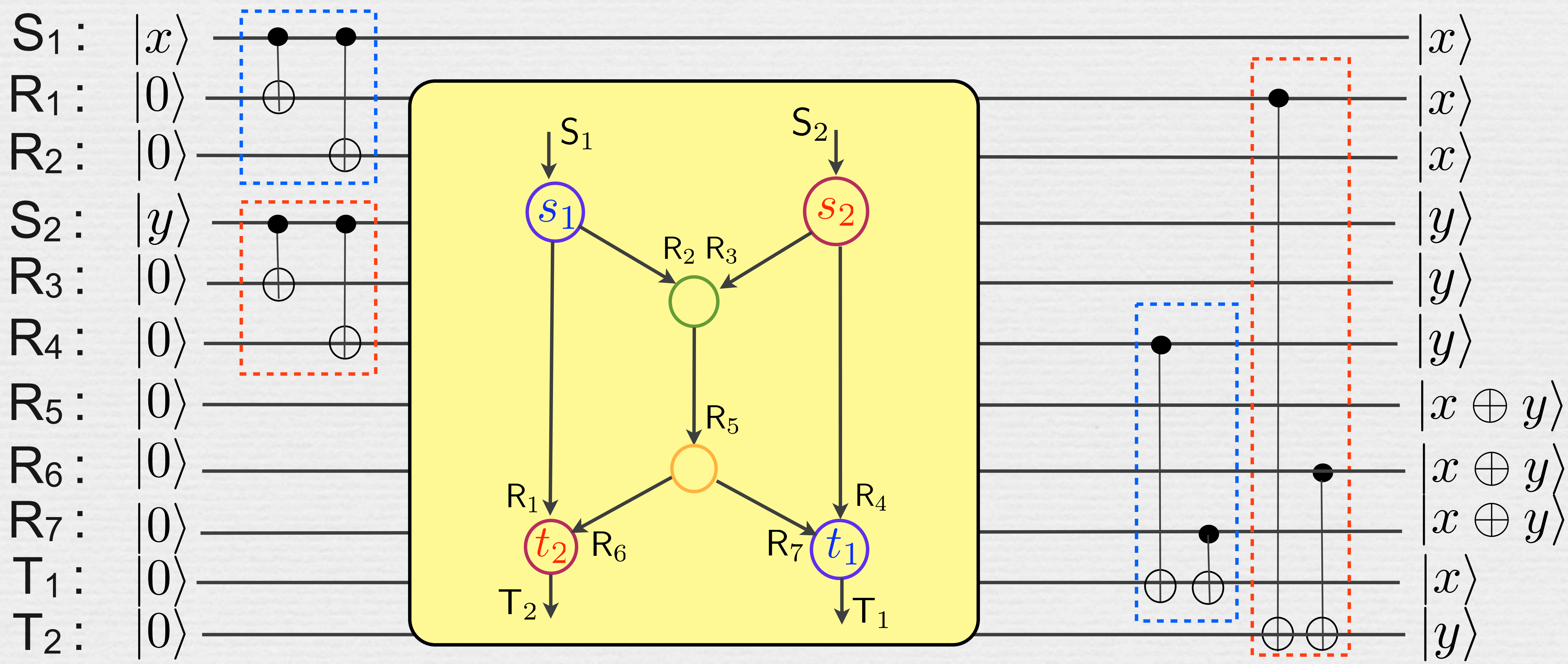
final state: $$\sum_{x,y\in\{0,1\}} \alpha_{xy}|x\rangle_{\mathsf{S}_1}|x\rangle_{\mathsf{R}_1}|x\rangle_{\mathsf{R}_2}|y\rangle_{\mathsf{S}_2}|y\rangle_{\mathsf{R}_3}|y\rangle_{\mathsf{R}_4}\otimes$$
$$|x\oplus y\rangle_{\mathsf{R}_5}|x\oplus y\rangle_{\mathsf{R}_6}|x\oplus y\rangle_{\mathsf{R}_7}|x\rangle_{\mathsf{T}_1}|y\rangle_{\mathsf{T}_2}$$



initial state: $\sum_{x,y\in\{0,1\}} \alpha_{xy}|x\rangle_{\mathsf{S}_1}|y\rangle_{\mathsf{S}_2}$

final state: $\sum_{x,y\in\{0,1\}} \alpha_{xy}|x\rangle_{\mathsf{S}_1}|x\rangle_{\mathsf{R}_1}|x\rangle_{\mathsf{R}_2}|y\rangle_{\mathsf{S}_2}|y\rangle_{\mathsf{R}_3}|y\rangle_{\mathsf{R}_4}\otimes$
$|x\oplus y\rangle_{\mathsf{R}_5}|x\oplus y\rangle_{\mathsf{R}_6}|x\oplus y\rangle_{\mathsf{R}_7}|x\rangle_{\mathsf{T}_1}|y\rangle_{\mathsf{T}_2}$

ideal state: $\sum_{x,y\in\{0,1\}} \alpha_{xy}|x\rangle_{\mathsf{T}_1}|y\rangle_{\mathsf{T}_2}$

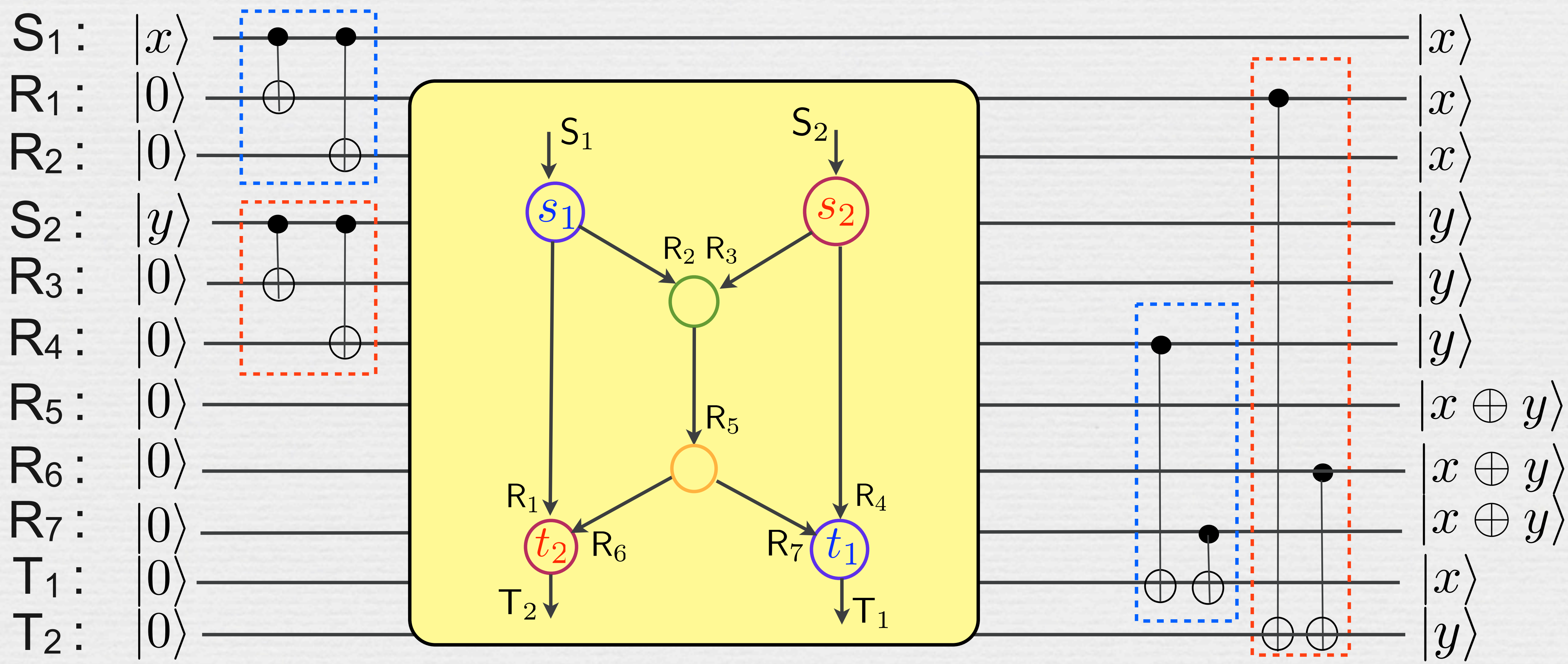


initial state: $\sum_{x,y\in\{0,1\}} \alpha_{xy}|x\rangle_{\mathsf{S}_1}|y\rangle_{\mathsf{S}_2}$

need to be removed (disentangled)

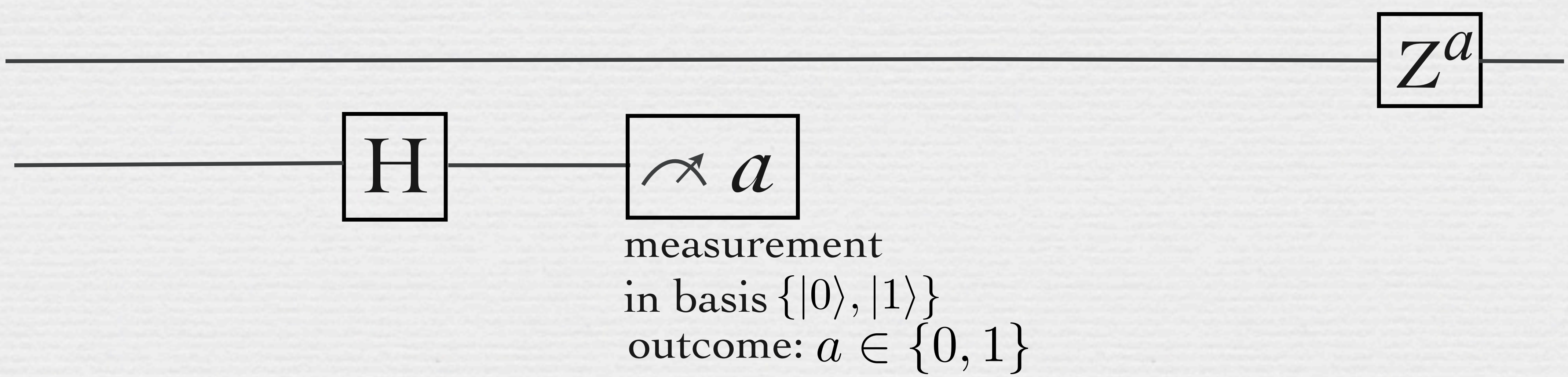
final state: $\sum_{x,y\in\{0,1\}} \alpha_{xy}\boxed{|x\rangle_{\mathsf{S}_1}|x\rangle_{\mathsf{R}_1}|x\rangle_{\mathsf{R}_2}|y\rangle_{\mathsf{S}_2}|y\rangle_{\mathsf{R}_3}|y\rangle_{\mathsf{R}_4}\otimes |x\oplus y\rangle_{\mathsf{R}_5}|x\oplus y\rangle_{\mathsf{R}_6}|x\oplus y\rangle_{\mathsf{R}_7}}|x\rangle_{\mathsf{T}_1}|y\rangle_{\mathsf{T}_2}$

ideal state: $\sum_{x,y\in\{0,1\}} \alpha_{xy}|x\rangle_{\mathsf{T}_1}|y\rangle_{\mathsf{T}_2}$



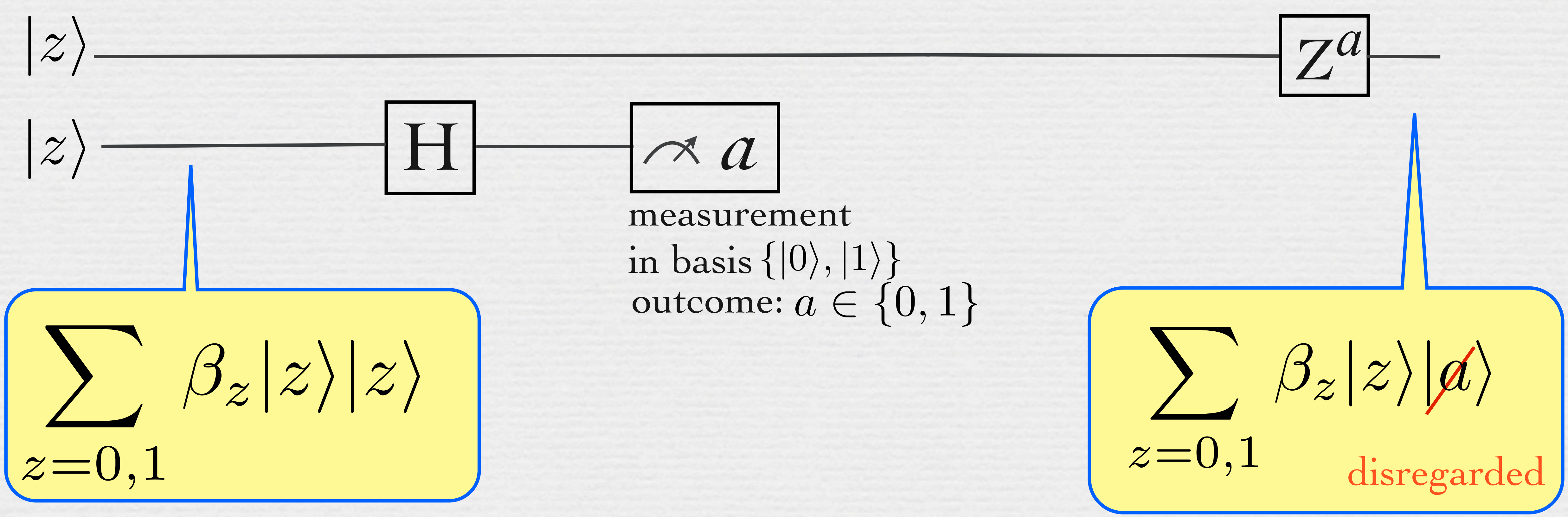
II. removal of internal registers

A TRICK:



II. removal of internal registers

A TRICK:

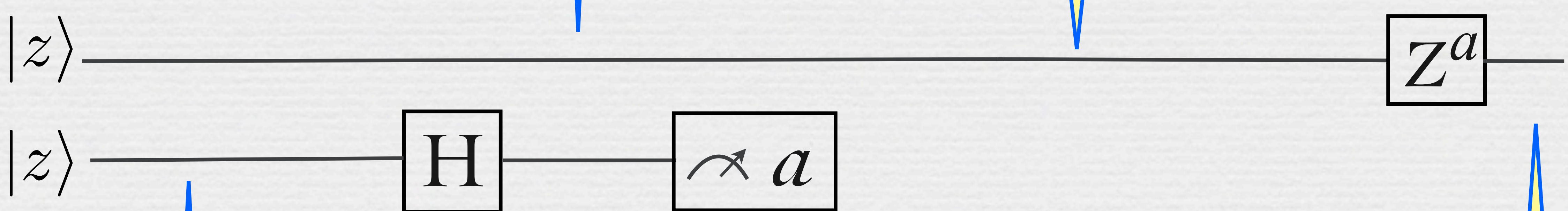


II. removal of internal registers

A TRICK:

$$\sum_{z=0,1} \beta_z |z\rangle \frac{|0\rangle + (-1)^z |1\rangle}{\sqrt{2}}$$

$$\sum_{z=0,1} \beta_z |z\rangle|0\rangle \text{ if } a = 0$$
$$\sum_{z=0,1} \beta_z (-1)^z |z\rangle|1\rangle \text{ if } a = 1$$



measurement
in basis $\{|0\rangle, |1\rangle\}$
outcome: $a \in \{0, 1\}$

$$\sum_{z=0,1} \beta_z |z\rangle|z\rangle$$

$$\sum_{z=0,1} \beta_z |z\rangle|\cancel{a}\rangle$$

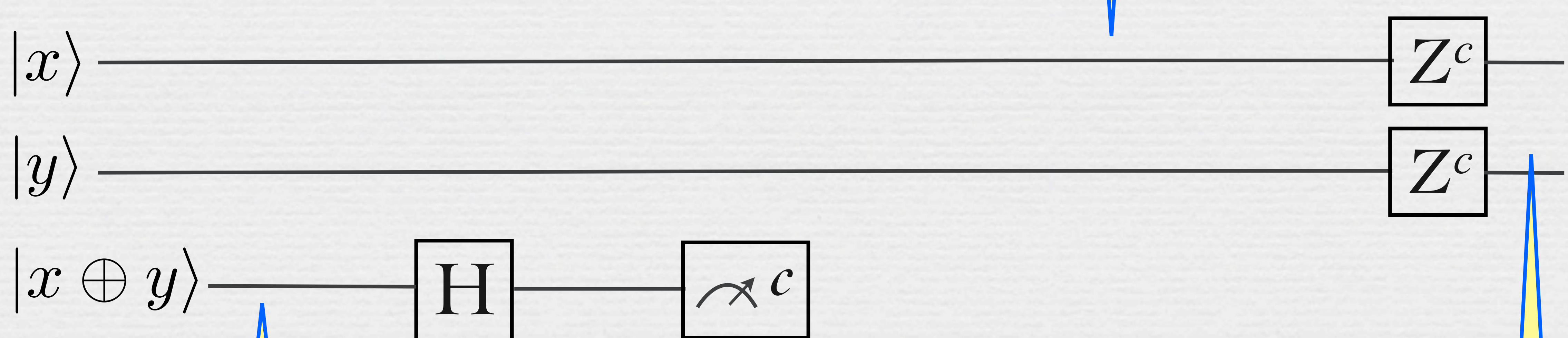
disregarded

II. removal of internal registers

ANOTHER TRICK:

$$\sum_{x,y=0,1} \gamma_{xy}|x\rangle|y\rangle|0\rangle \text{ if } c=0$$

$$\sum_{x,y=0,1} \gamma_{xy}(-1)^{x\oplus y}|x\rangle|y\rangle|1\rangle \text{ if } c=1$$

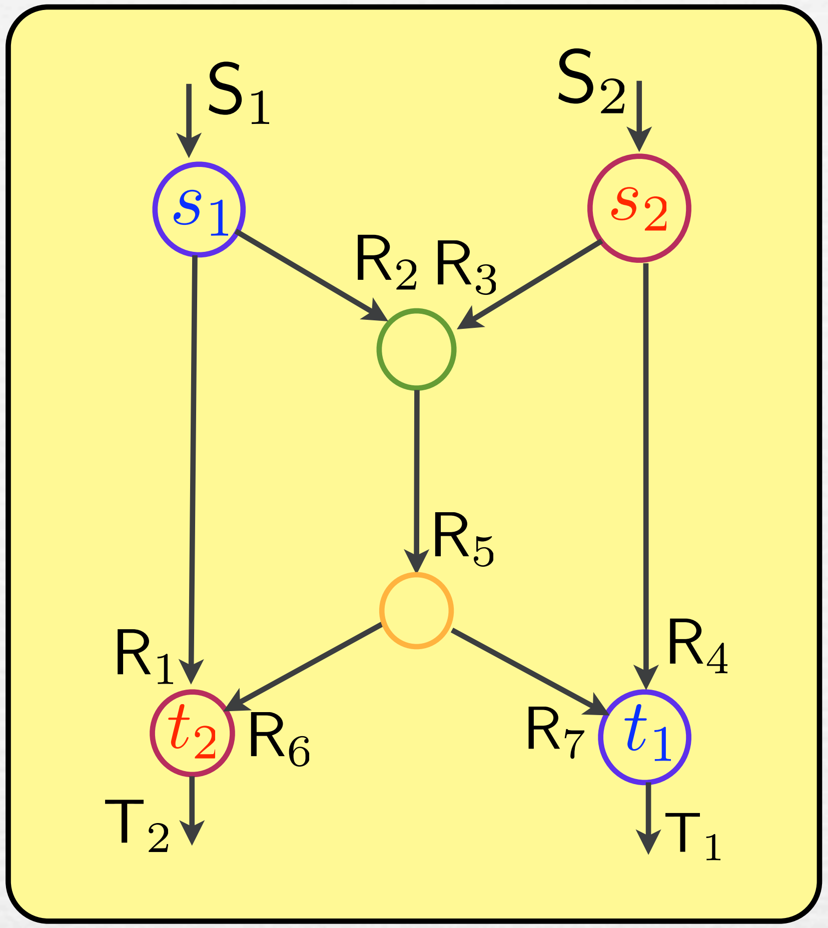


$$\sum_{x,y=0,1} \gamma_{xy}|x\rangle|y\rangle|x\oplus y\rangle$$

$$\sum_{x,y=0,1} \gamma_{xy}|x\rangle|y\rangle|\cancel{c}\rangle$$

II. removal of internal registers

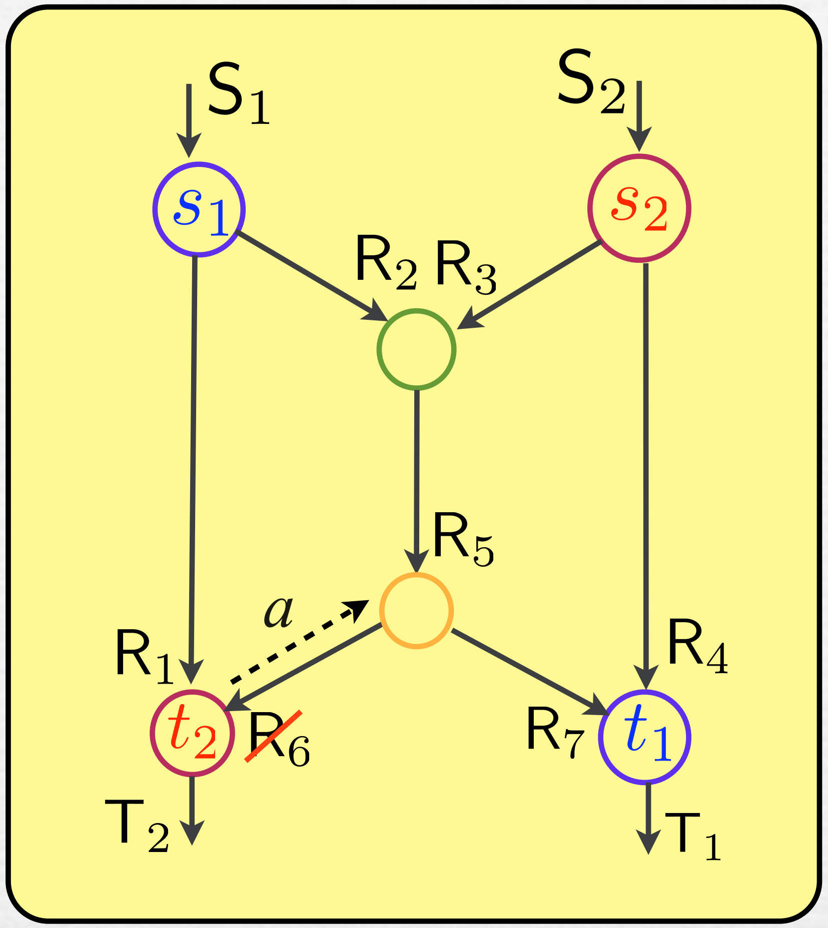
idea: phases can always be corrected at the previous node



$S_1 : |x\rangle$
$R_1 : |x\rangle$
$R_2 : |x\rangle$
$S_2 : |y\rangle$
$R_3 : |y\rangle$
$R_4 : |y\rangle$
$R_5 : |x \oplus y\rangle$
$R_6 : |x \oplus y\rangle$
$R_7 : |x \oplus y\rangle$
$T_1 : |x\rangle$
$T_2 : |y\rangle$

II. removal of internal registers

idea: phases can always be corrected at the previous node



: 1 bit

$$\sum_{x,y\in\{0,1\}} \alpha_{xy}|x\rangle_{\mathsf{S}_1}|x\rangle_{\mathsf{R}_1}|x\rangle_{\mathsf{R}_2}|y\rangle_{\mathsf{S}_2}|y\rangle_{\mathsf{R}_3}|y\rangle_{\mathsf{R}_4}\otimes$$

$$|x\oplus y\rangle_{\mathsf{R}_5}|\cancel{a}\rangle_{\mathsf{R}_6}|x\oplus y\rangle_{\mathsf{R}_7}|x\rangle_{\mathsf{T}_1}|y\rangle_{\mathsf{T}_2}$$

$\mathsf{S}_1 : |x\rangle$

$\mathsf{R}_1 : |x\rangle$

$\mathsf{R}_2 : |x\rangle$

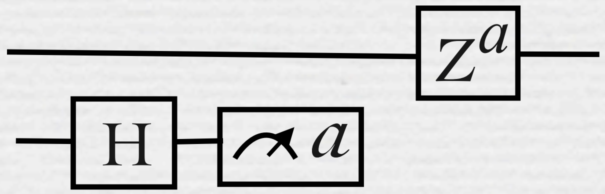
$\mathsf{S}_2 : |y\rangle$

$\mathsf{R}_3 : |y\rangle$

$\mathsf{R}_4 : |y\rangle$

$\mathsf{R}_5 : |x\oplus y\rangle$

$\mathsf{R}_6 : |x\oplus y\rangle$



$\mathsf{R}_7 : |x\oplus y\rangle$

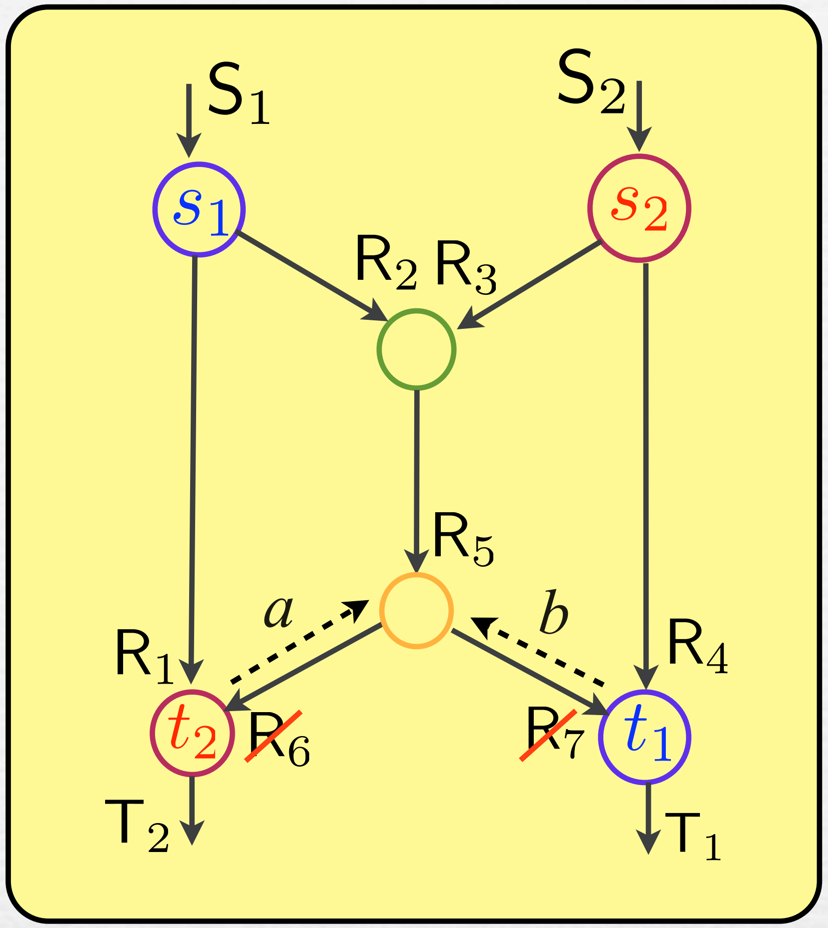
$\mathsf{T}_1 : |x\rangle$

$\mathsf{T}_2 : |y\rangle$

II. removal of internal registers

idea: phases can always be corrected at the previous node

$\dashrightarrow$: 1 bit



$$\sum_{x,y\in\{0,1\}} \alpha_{xy}|x\rangle_{S_1}|x\rangle_{R_1}|x\rangle_{R_2}|y\rangle_{S_2}|y\rangle_{R_3}|y\rangle_{R_4}\otimes$$
$$|x\oplus y\rangle_{R_5}|\cancel{a}\rangle_{R_6}|x\oplus y\rangle_{R_7}|x\rangle_{T_1}|y\rangle_{T_2}$$

$S_1 : |x\rangle$

$R_1 : |x\rangle$

$R_2 : |x\rangle$

$S_2 : |y\rangle$

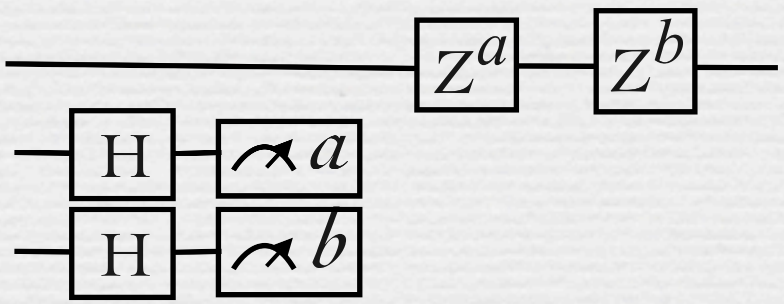
$R_3 : |y\rangle$

$R_4 : |y\rangle$

$R_5 : |x\oplus y\rangle$

$R_6 : |x\oplus y\rangle$

$R_7 : |x\oplus y\rangle$



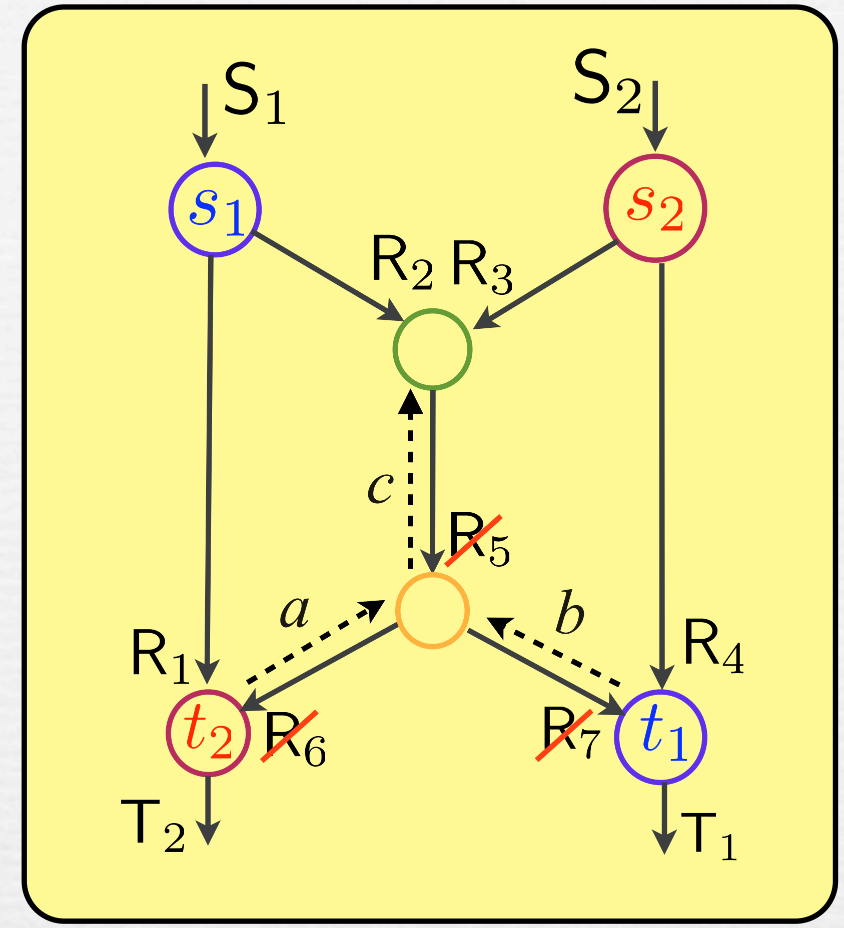
$T_1 : |x\rangle$

$T_2 : |y\rangle$

II. removal of internal registers

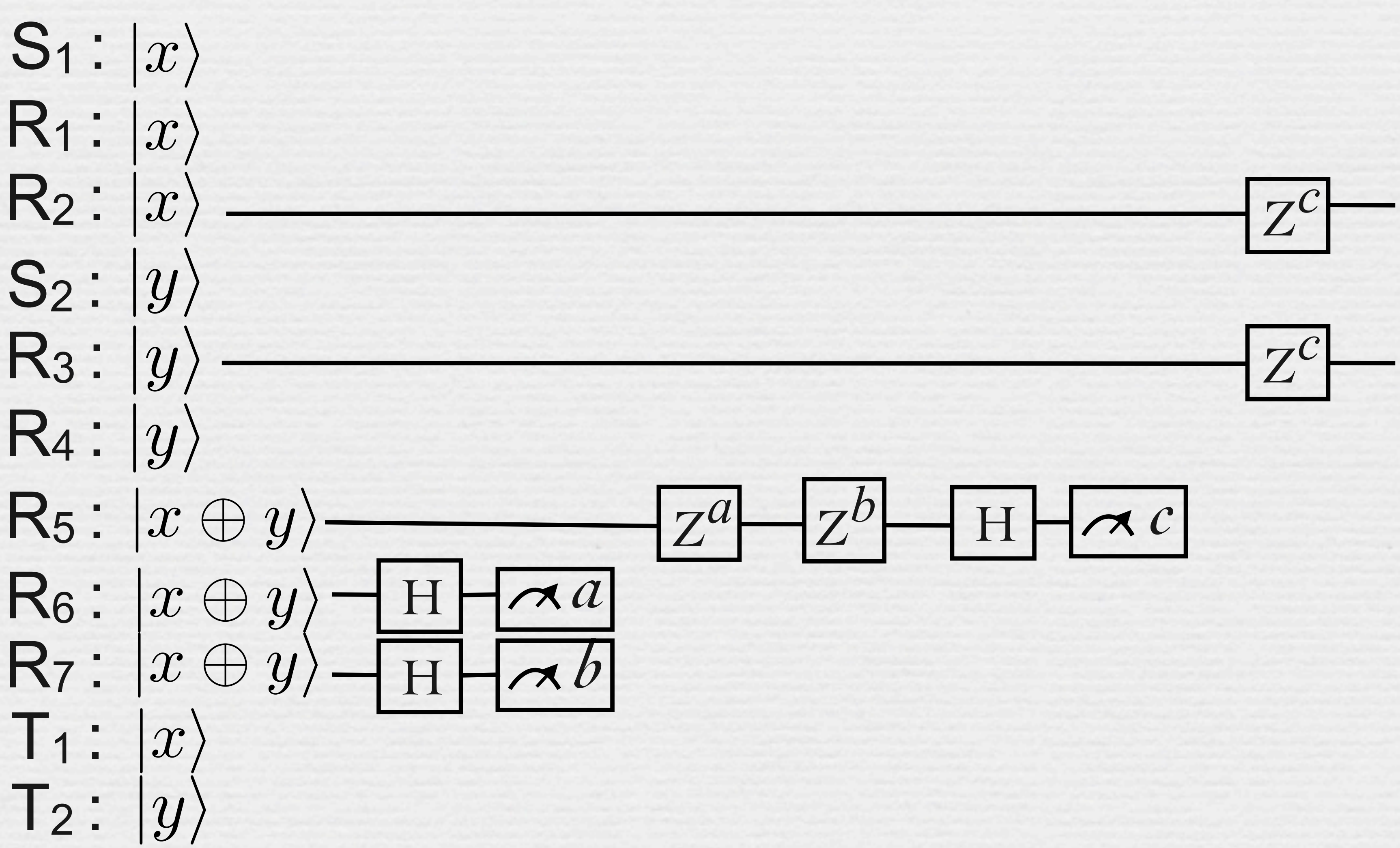
idea: phases can always be corrected at the previous node

: 1 bit



$$\sum_{x,y\in\{0,1\}} \alpha_{xy}|x\rangle_{S_1}|x\rangle_{R_1}|x\rangle_{R_2}|y\rangle_{S_2}|y\rangle_{R_3}|y\rangle_{R_4}\otimes$$

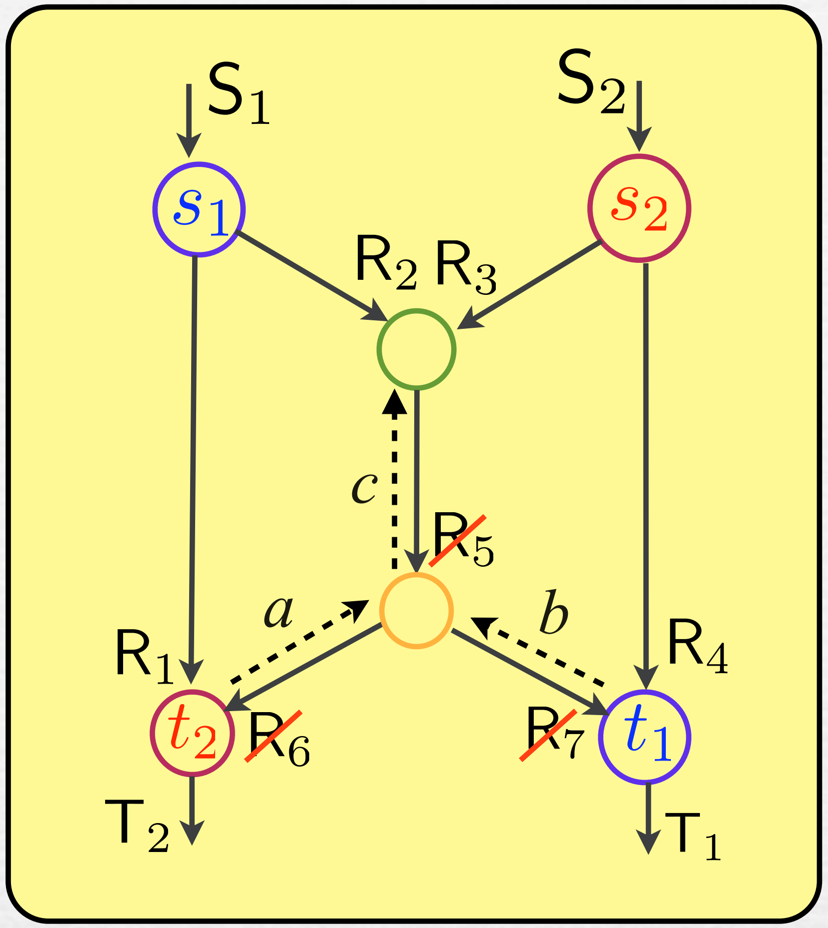
$$|x\oplus y\rangle_{R_5}|\cancel{a}\rangle_{R_6}|x\oplus y\rangle_{R_7}|x\rangle_{T_1}|y\rangle_{T_2}$$



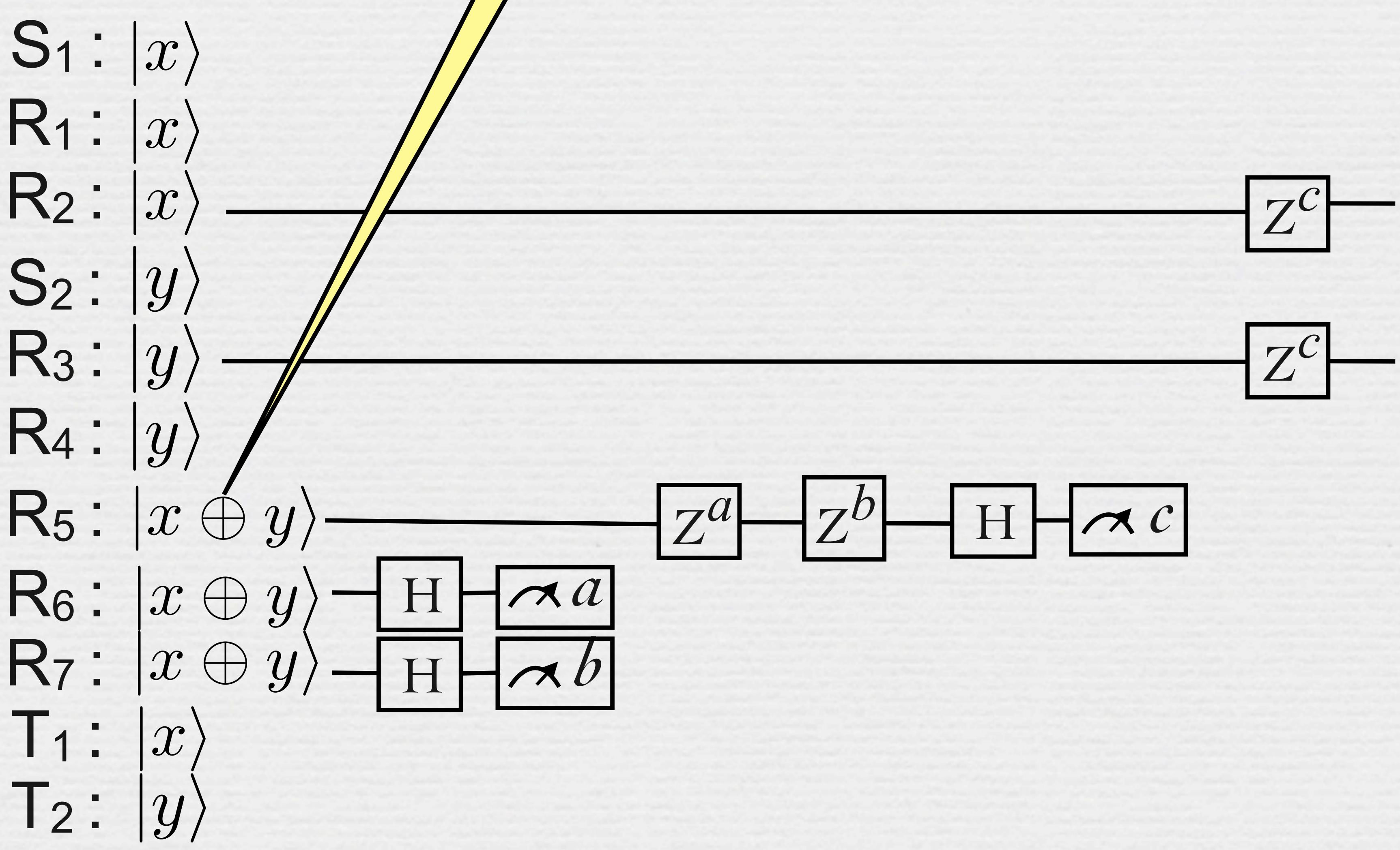
II. removal of internal registers

idea: phases can always be corrected at the previous node

⇢ : 1 bit



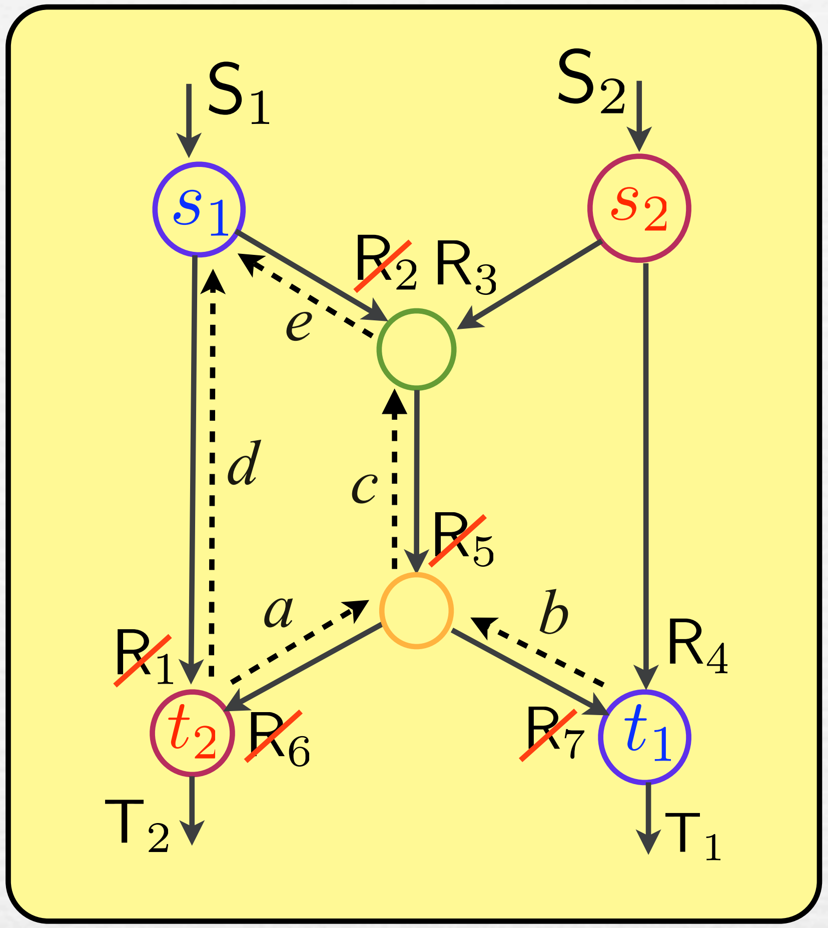
R_5 was created using R_2 and R_3



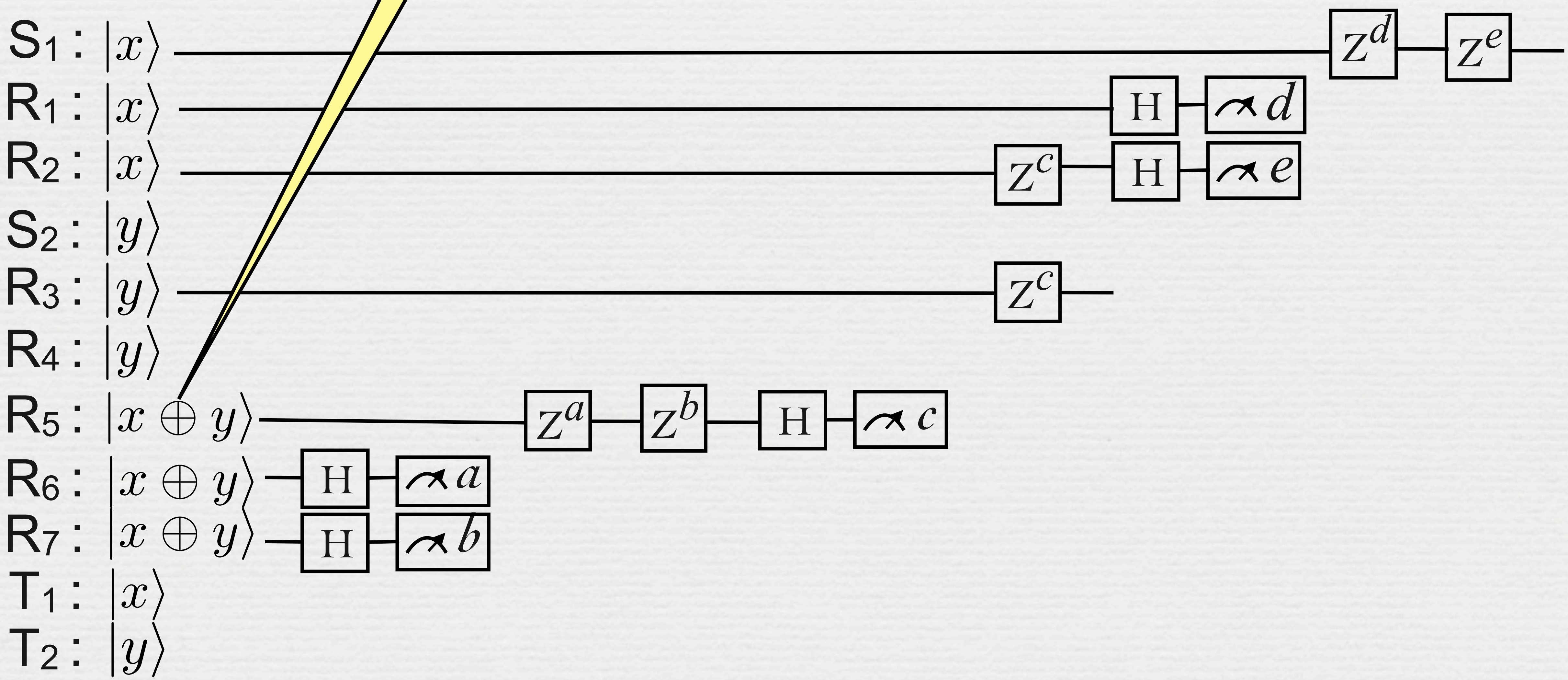
II. removal of internal registers

idea: phases can always be corrected at the previous node

: 1 bit

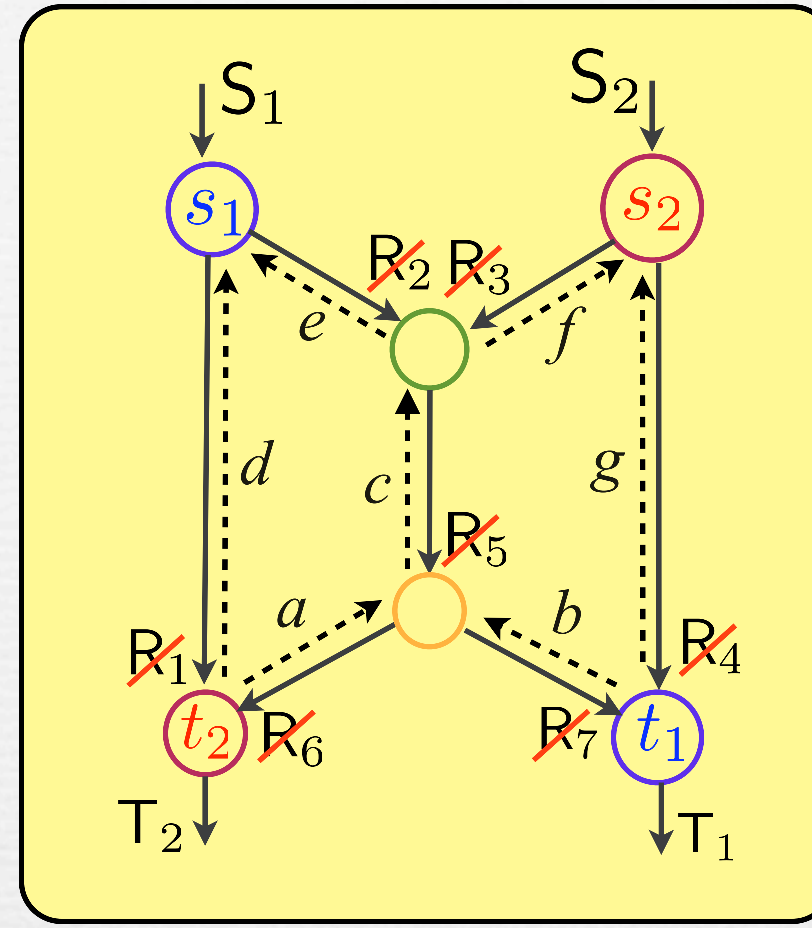


R_5 was created using R_2 and R_3



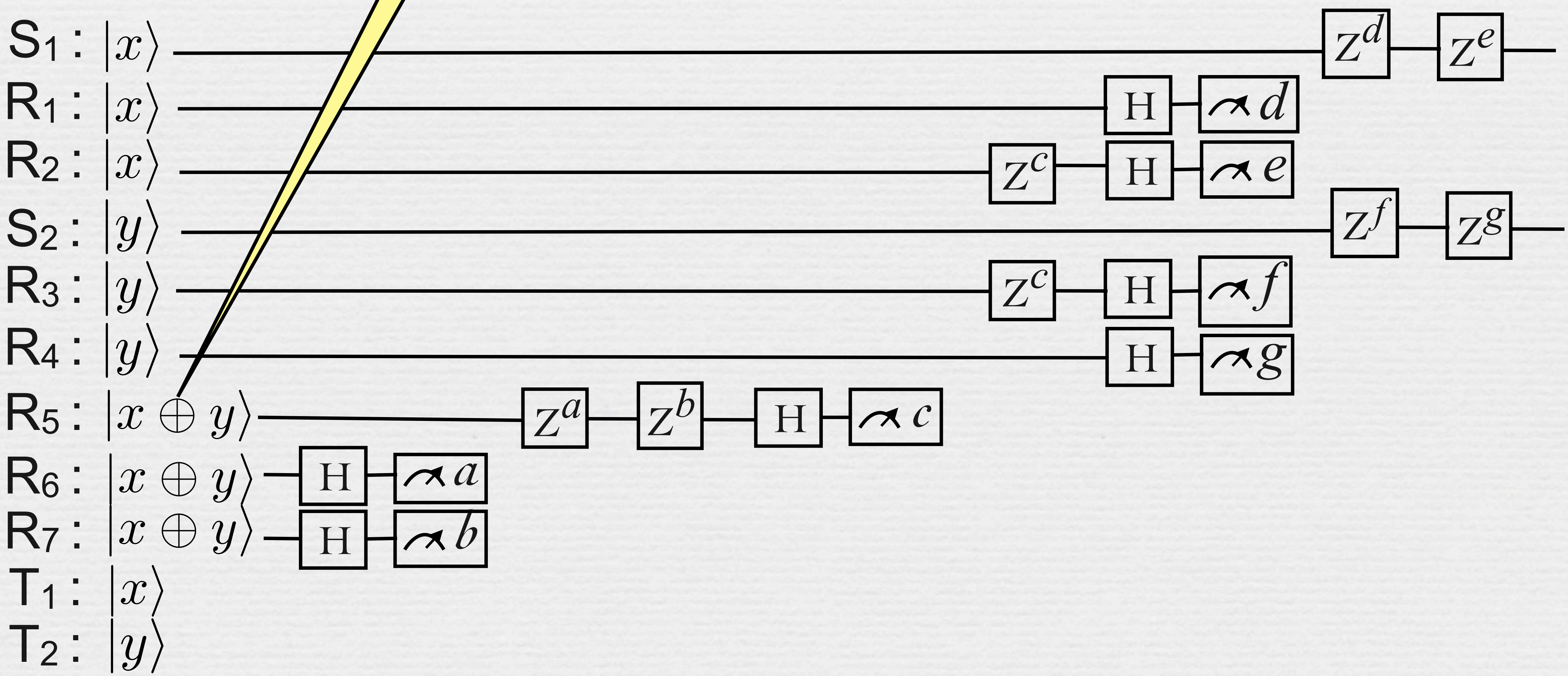
II. removal of internal registers

idea: phases can always be corrected at the previous node

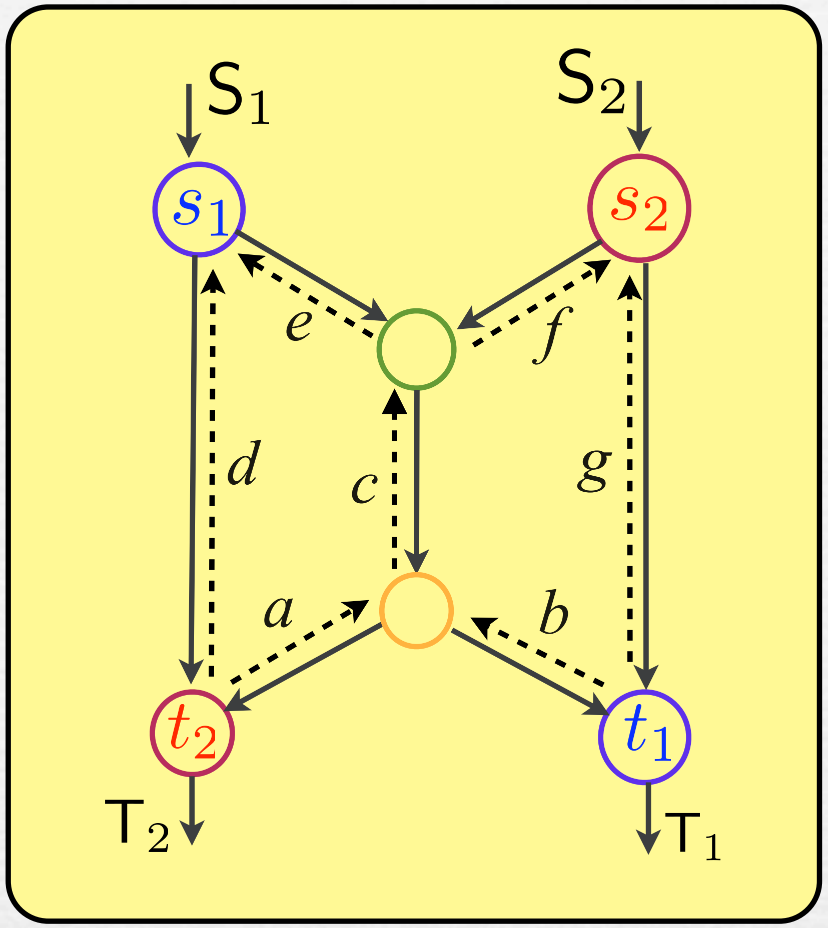


$\dashrightarrow$: 1 bit

R_5 was created using R_2 and R_3



III. removal of initial registers



current state: $$\sum_{x,y\in\{0,1\}} \alpha_{xy} \boxed{|x\rangle_{\mathsf{S}_1} |y\rangle_{\mathsf{S}_2}} |x\rangle_{\mathsf{T}_1} |y\rangle_{\mathsf{T}_2}$$

ideal state: $$\sum_{x,y\in\{0,1\}} \alpha_{xy} |x\rangle_{\mathsf{T}_1} |y\rangle_{\mathsf{T}_2}$$

$\mathsf{S}_1 : |x\rangle$

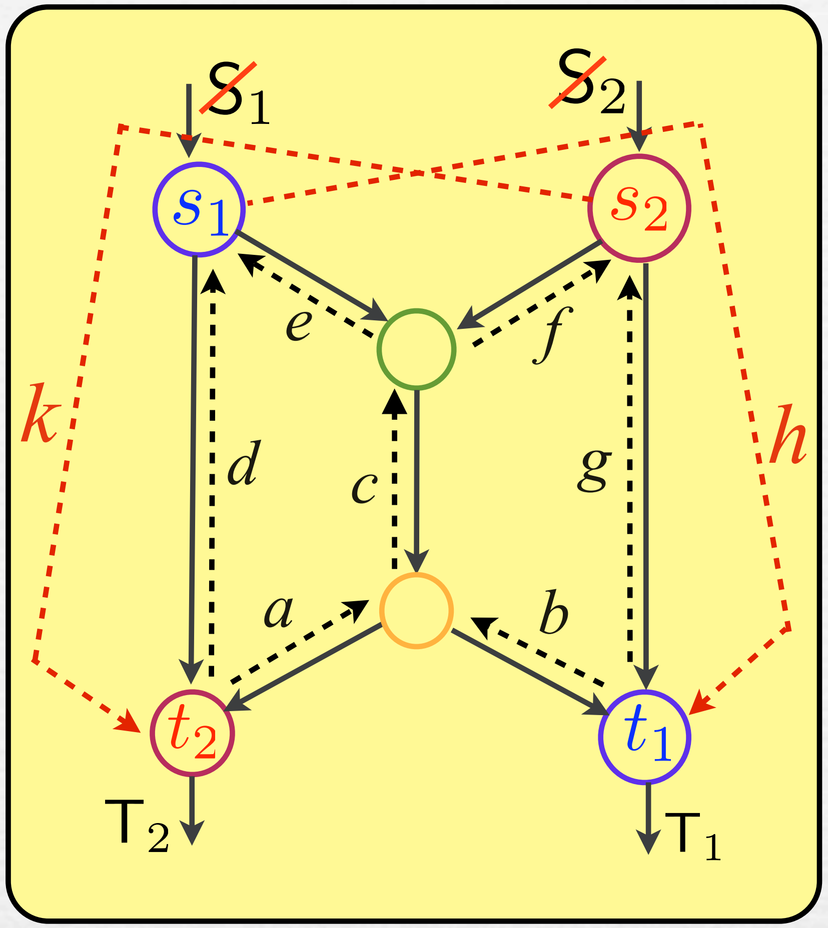
$\mathsf{S}_2 : |y\rangle$

$\mathsf{T}_1 : |x\rangle$

$\mathsf{T}_2 : |y\rangle$

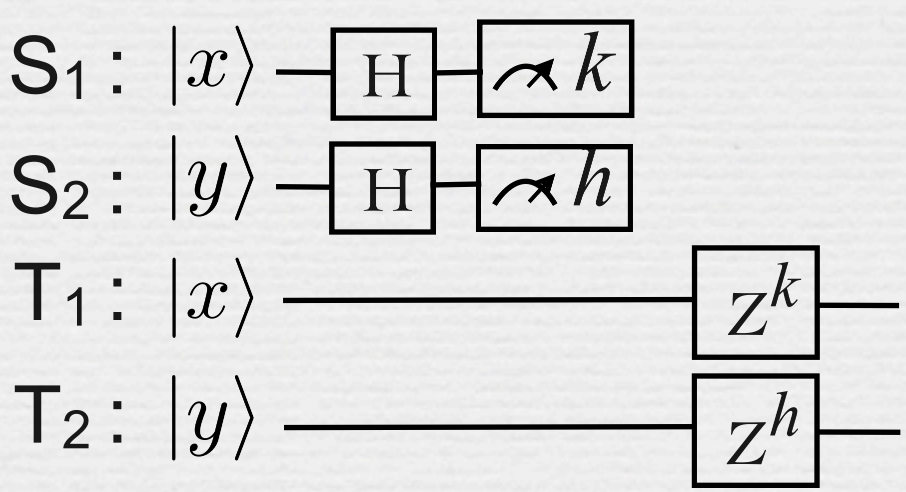
III. removal of initial registers

exception: phases are corrected at the target nodes



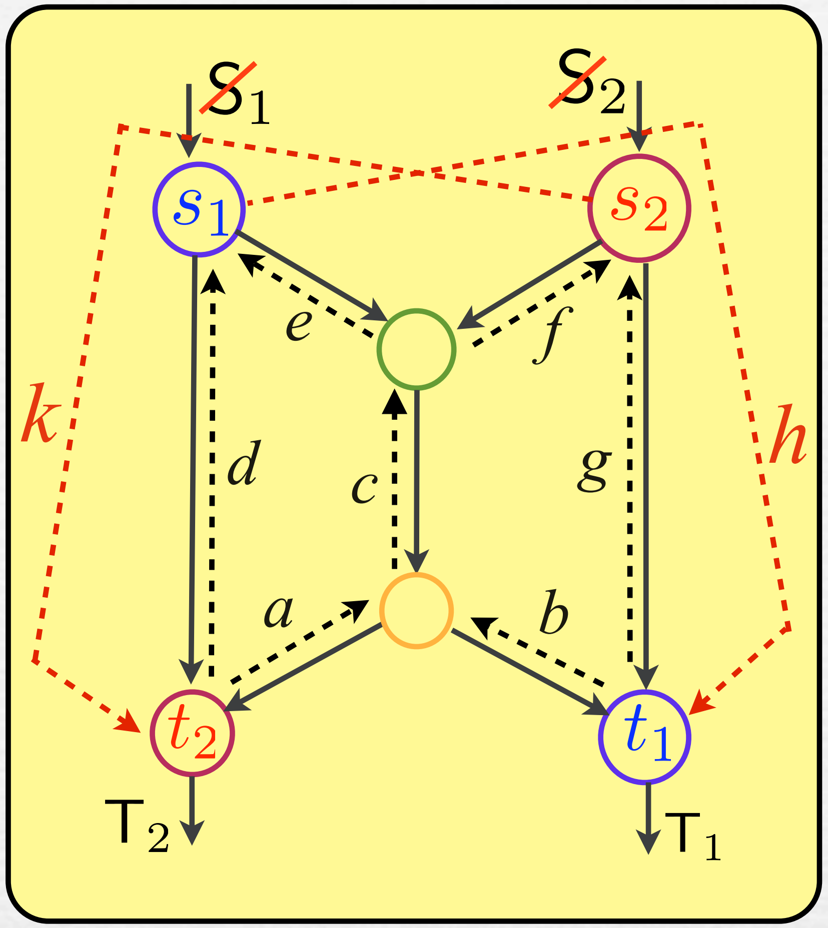
current state: $$\sum_{x,y\in\{0,1\}} \alpha_{xy} \boxed{|x\rangle_{\mathsf{S}_1}|y\rangle_{\mathsf{S}_2}} |x\rangle_{\mathsf{T}_1}|y\rangle_{\mathsf{T}_2}$$

ideal state: $$\sum_{x,y\in\{0,1\}} \alpha_{xy} |x\rangle_{\mathsf{T}_1}|y\rangle_{\mathsf{T}_2}$$



III. removal of initial registers

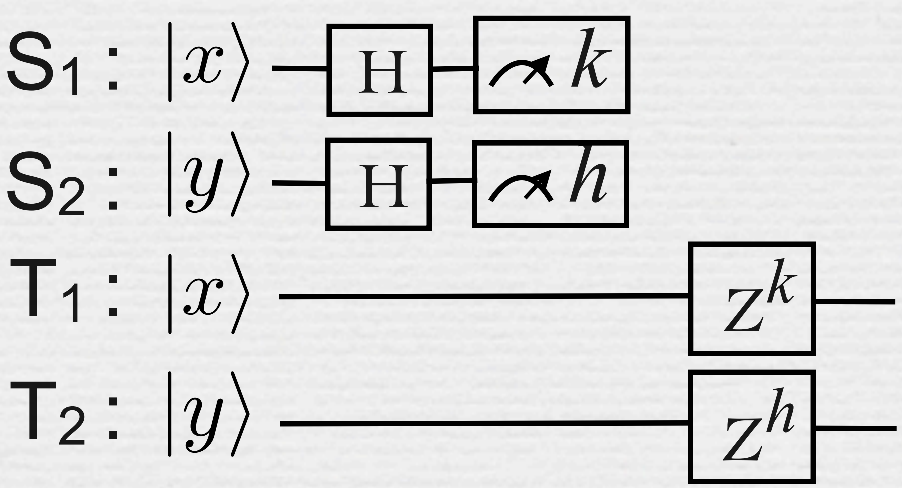
exception: phases are corrected at the target nodes



current state: $$\sum_{x,y\in\{0,1\}} \alpha_{xy} \boxed{|x\rangle_{\mathsf{S}_1}|y\rangle_{\mathsf{S}_2}} |x\rangle_{\mathsf{T}_1}|y\rangle_{\mathsf{T}_2}$$

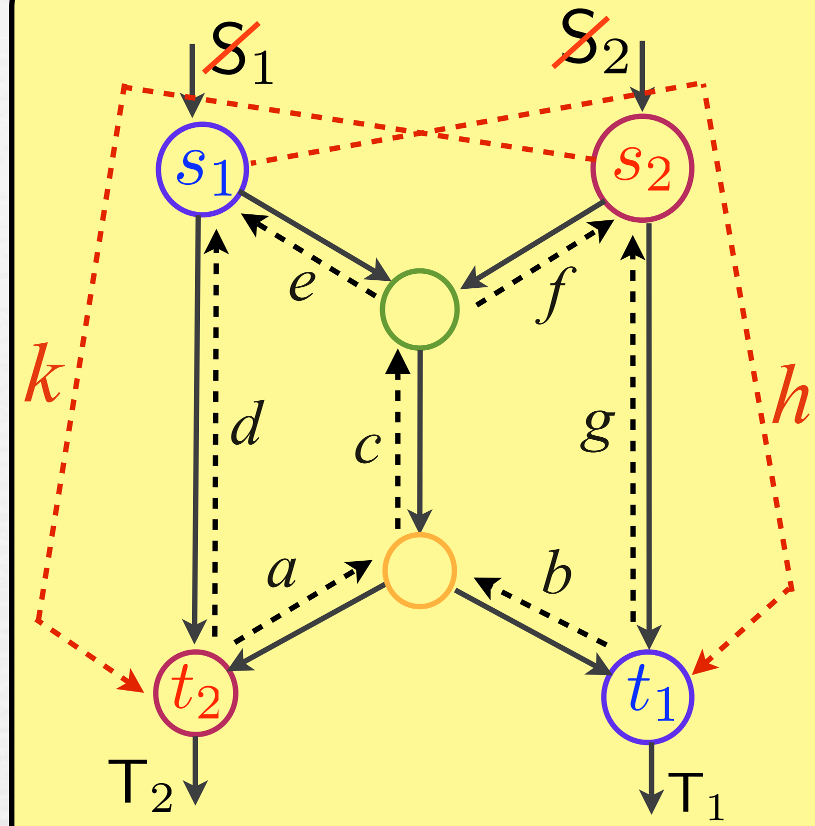
$$\Downarrow$$

ideal state: $$\sum_{x,y\in\{0,1\}} \alpha_{xy} |x\rangle_{\mathsf{T}_1}|y\rangle_{\mathsf{T}_2}$$

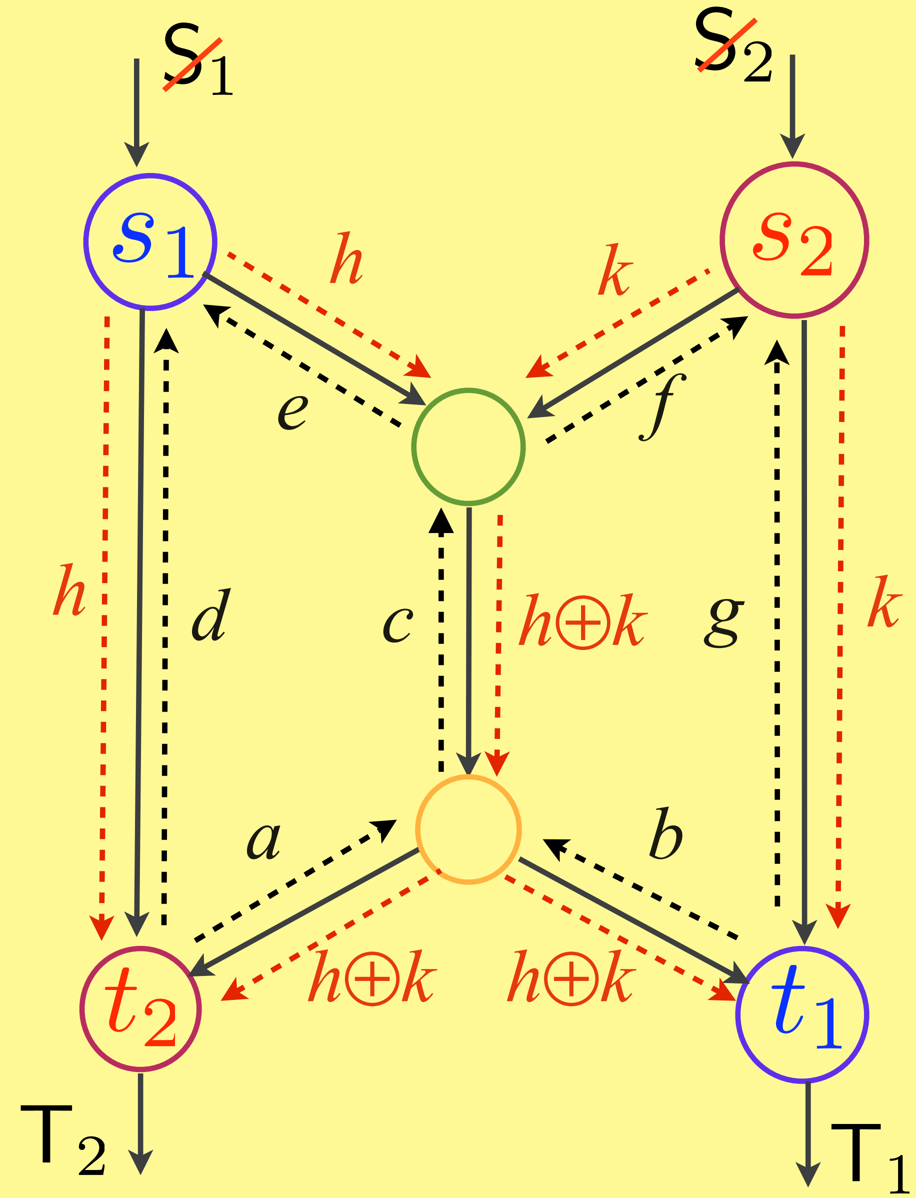


III. removal of initial registers

exception: phases are corrected at the target nodes



equivalently:



→ : 1 qubit (original direction)

⇢ : 1 bit (reverse direction)

⇢ : 1 bit (original direction)

<u>one qubit + two bits sent per edge</u>

Main Theorem (again)

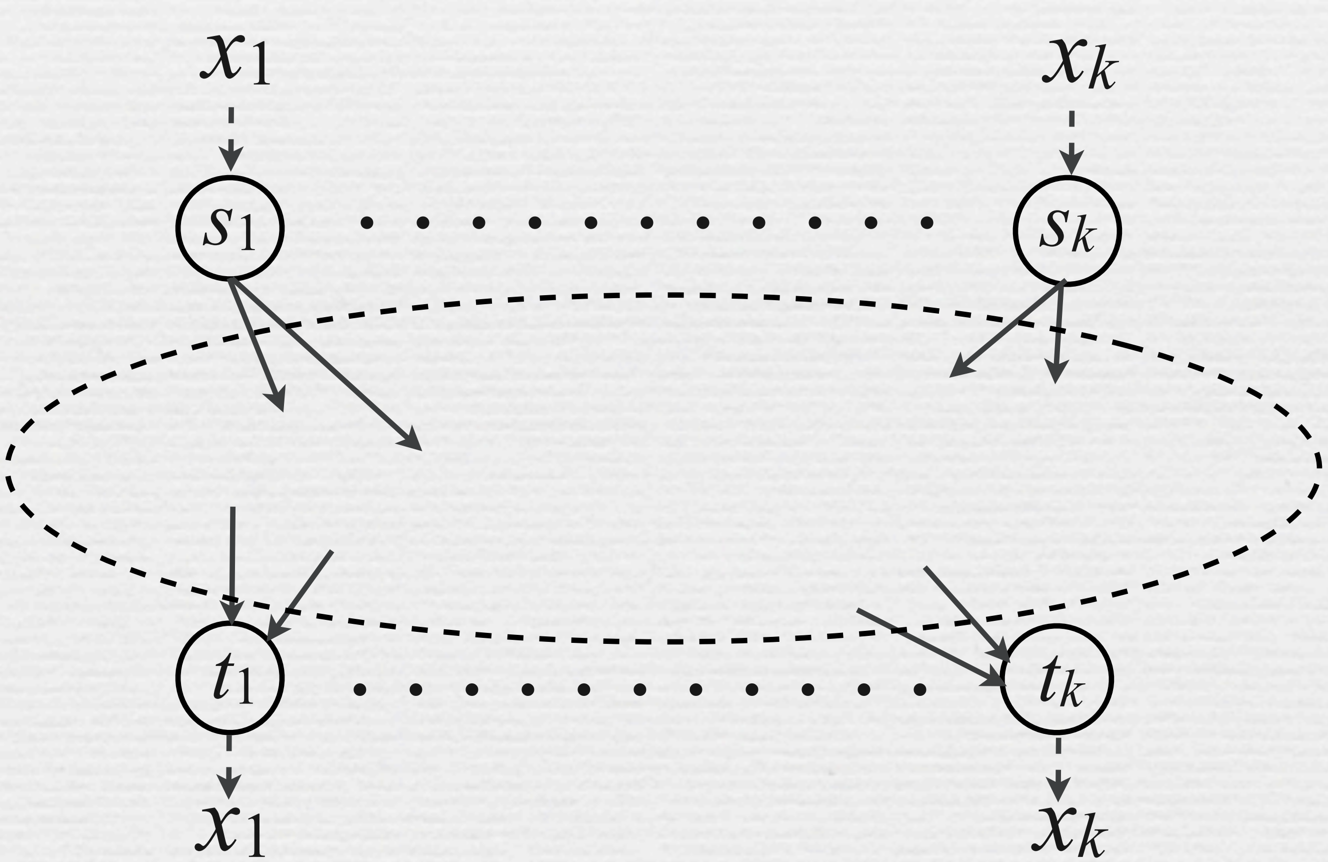
Main Theorem

Suppose that a given instance of the classical k-pair problem has a solution.
Then the associated quantum instance has a perfect solution if free classical communication is allowed.

one qubit + two bits sent per edge

General Quantum Protocol

<u>classical coding scheme</u>



<u>quantum simulation</u>

Three steps:

I. node-by-node simulation

II. removal of internal registers

III. removal of initial registers

Conclusions

- Without additional resources, perfect quantum network coding is impossible in general
- With free classical communication, perfect quantum coding is possible whenever classical coding is feasible
  - this works even for nonlinear classical schemes
  - at most two bits of classical communication are sent per edge
- Our proof is constructive: efficient construction of a quantum perfect transmission protocol from any classical coding scheme