Quantum query complexity of minor-closed graph properties

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Consider the basic computational task of deciding whether an n-vertex graph satisfies a given property. For most properties, even a randomized classical algorithm must make $\Omega(n^2)$ queries to the edges of the graph to answer the question. Quantum algorithms can often decide graph properties using dramatically fewer queries, but establishing the precise query complexity of a given property can be challenging. The quantum query complexities of a few specific graph properties have been determined, including connectivity [4], bipartiteness [9], and planarity [2]. However, a systematic understanding of the quantum query complexity of graph properties is lacking.

A natural type of graph property is the property of containing a fixed graph H as a subgraph. If H is the complete graph on 3 vertices, this is the triangle problem, which has been open for more than ten years [3]. One of the early applications of quantum walk search was an algorithm for the triangle problem using $\tilde{O}(n^{1.3})$ quantum queries [7]. While this was a significant step forward, the best known lower bound is only $\Omega(n)$. More generally, if H is a d-vertex subgraph for $d > 3$, the best known upper bound is $O(n^{2-2/d})$ [7], whereas the best known lower bound is again only linear.

In this work, we study quantum algorithms for a related family of graph properties, those that are *minor closed*. Graph minors are defined using an operation called edge contraction, in which two adjacent vertices are merged to form a single vertex (see Figure 1). We say that H is a minor of G if H can be obtained by contracting edges of a subgraph of G. By a celebrated result of Robertson and Seymour [8], minor-closed properties can be characterized by a finite set of excluded minors, so these properties are equivalent to deciding whether the input graph contains any of a certain set of graphs as a minor.

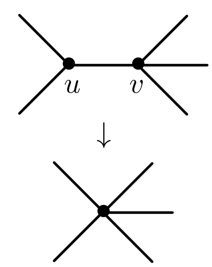

Figure 1: Contracting the edge (u, v).

Many natural properties of graphs are closed under taking minors. The best-known example is the property of being planar. Other examples include the property of being a forest, being embeddable on a fixed two-dimensional manifold, having treewidth at most k, or not containing a path of length ℓ. While it has been shown that planarity has quantum query complexity $\Theta(n^{3/2})$ [2], we are not aware of any other previous quantum algorithms or lower bounds for minor-closed properties.

In this work, we significantly improve our understanding of the quantum query complexity of minor-closed graph properties. In particular, we show that the complexity of such properties depends crucially on whether the property can be characterized by a finite set of forbidden subgraphs. We call such a property a *forbidden subgraph property* (FSP). First, consider minor-closed properties that are not FSP. This covers most minor-closed properties, including all aforementioned examples except the property of not containing a path of a given length. We tightly characterize the quantum query complexity of all such properties as $\Theta(n^{3/2})$. Second, we show that all minor-closed properties that *are* FSP can be solved strictly faster, in $O(n^\alpha)$ queries for some $\alpha < 3/2$ that may depend on the property.

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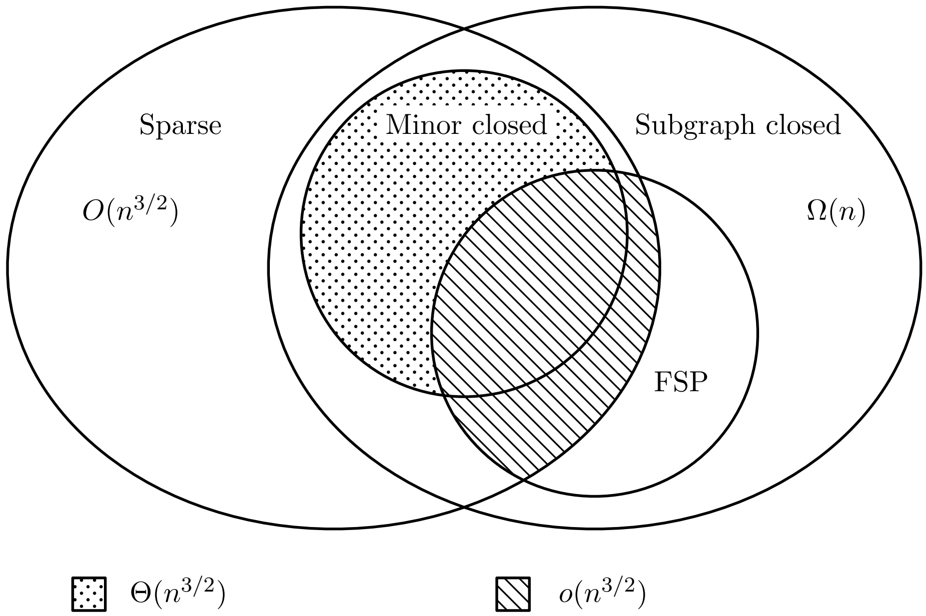

Figure 2: Summary of the main results.

Figure 2 summarizes our main results. All sparse graph properties—those that can only hold if the graph has $O(n)$ edges, which include minor-closed properties as a special case—have an easy upper bound of $O(n^{3/2})$. Our first main contribution is a matching lower bound of $\Omega(n^{3/2})$ for any minor-closed property that is not FSP. Our second main contribution is a quantum algorithm for any sparse graph property that is FSP using $o(n^{3/2})$ quantum queries.

To prove our lower bound on minor-closed properties that are not FSP, we use the quantum adversary method [1]. The basic approach to identifying hard-to-distinguish positive and negative instances is similar to the quantum lower bound for connectivity [4]. However, establishing the existence of these instances using only the hypothesis that the property is minor-closed and not FSP is technically challenging. Our proof relies on a detailed analysis of the structure of minor-closed properties with respect to forbidden topological minors and forbidden subgraphs.

Our algorithms for minor-closed properties that are FSP are a novel application of the quantum walk search framework [6]. The approach differs from previous applications of this formalism in several respects. Our walk makes essential use of the sparsity of the input graph, and in particular, requires queries even to identify which vertices of the graph to search over. Furthermore, the search quantizes a Markov chain in which different transitions occur at different rates; we optimize the performance of the algorithm by carefully choosing these rates.

This approach can be applied to find subgraphs of a sparse input graph more generally, even if the corresponding problem is not necessarily minor closed. In general, we show that $\tilde{O}(n^{\frac{3}{2}-\frac{1}{\mathrm{vc}(H)+1}})$ queries suffice to decide whether a sparse graph contains H has a subgraph, where $\mathrm{vc}(H)$ is the size of a minimal vertex cover of H. Furthermore, we describe several improved algorithms that take particular features of H into account. For example, we show that a path of length ℓ can be found in $\tilde{O}(n)$ queries for $\ell \in \{1, 2, 3, 4\}$; we also give nontrivial algorithms for longer paths.

Our quantum walk search algorithms can also exploit relaxed notions of sparsity. Using a result from extremal graph theory on the density of graphs that exclude a complete bipartite subgraph [5], this leads to further algorithms for subgraph finding. For example, we show that any bipartite d-

vertex subgraph H can be detected in $\tilde{O}(n^{2-\frac{1}{d}-\frac{2}{d+2}})$ quantum queries, substantially outperforming the algorithm of [7].

Finally, we show that a cycle on 4 vertices can be detected in only $\tilde{O}(n^{1.25})$ quantum queries. This may seem unexpected, since finding a 4-cycle is a natural generalization of triangle finding to a larger subgraph. Indeed, the previous best known algorithm for finding 4-cycles used $\tilde{O}(n^{1.5})$ queries [3], more than the $\tilde{O}(n^{1.3})$ queries for triangle finding. Our improvement shows that 4-cycles can be found in fewer quantum queries than in the best known quantum algorithm for finding 3-cycles.

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