Information propagation for interacting particle systems

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joint work with Sarah Harrison, Tobias Osborne, and Jens Eisert

Introduction

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  → Obvious answer: **Relativity $\Rightarrow$ No faster than the speed of light!**
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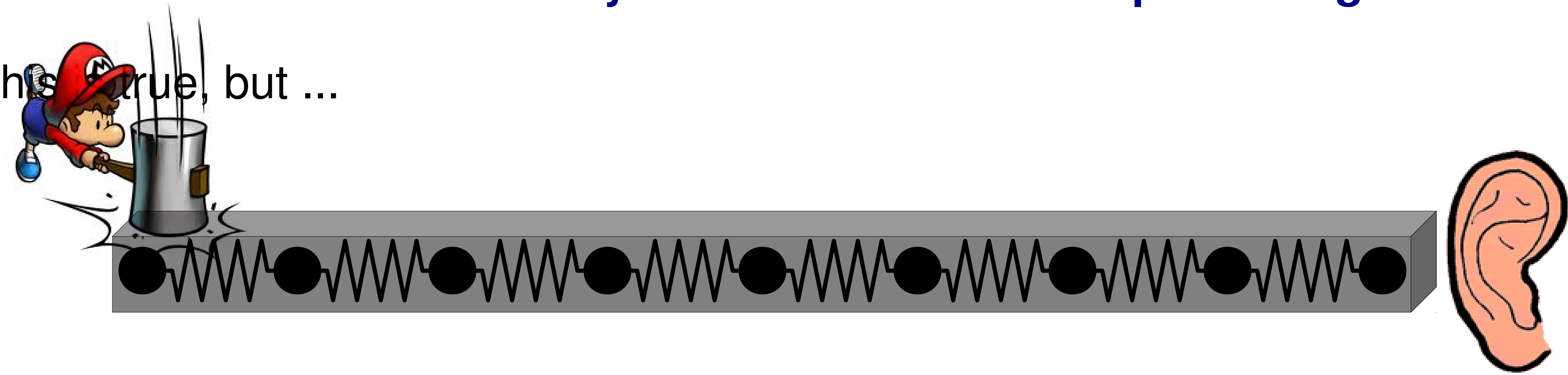

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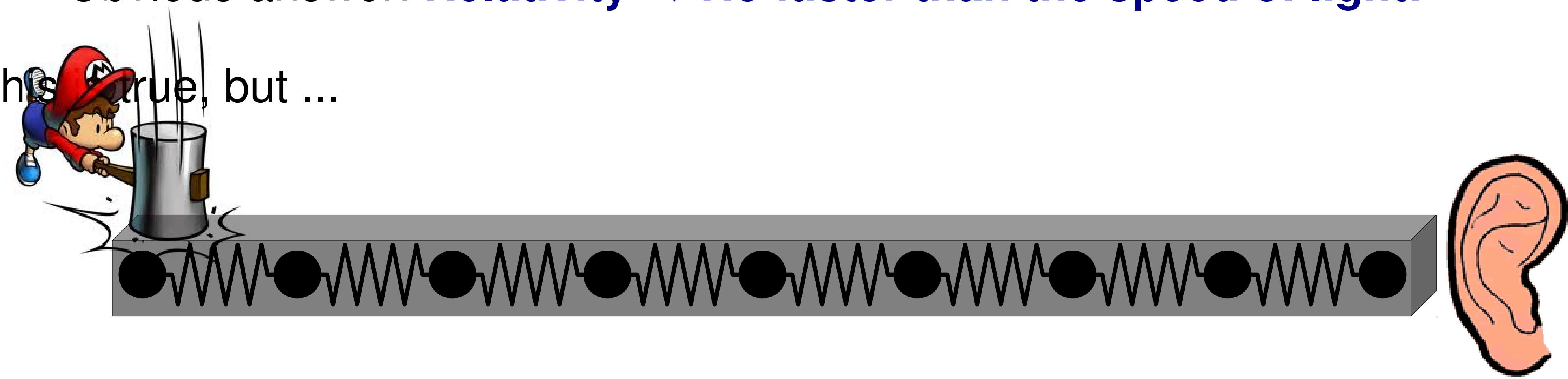

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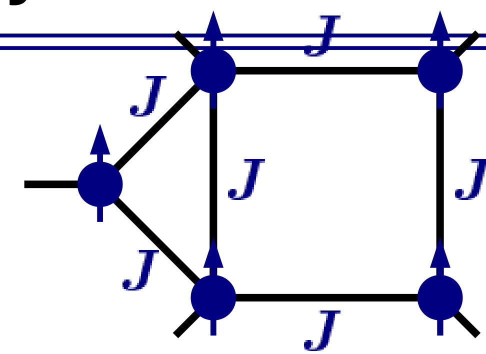
⇒ Finite propagation speed can be understood non-relativistically!

Quantum mechanical systems



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$$H = \sum_{<j,k>} h_{jk} \; ; \quad \|h_{jk}\|_{\mathrm{op}} \leq J$$: local Hamiltonian of bounded strength

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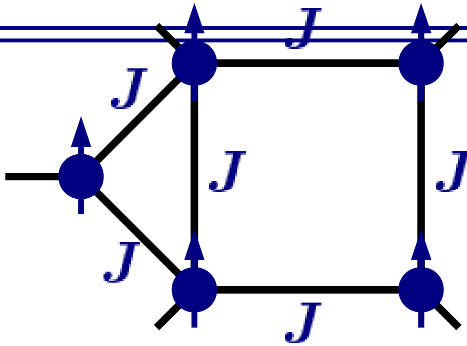

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[Lieb & Robinson '72, Hastings '04, Nachtergaele & Sims '06]

$$\|[A(t), B]\| \leq c\|A\|\|B\| \exp[-(L - vt)/\xi]$$

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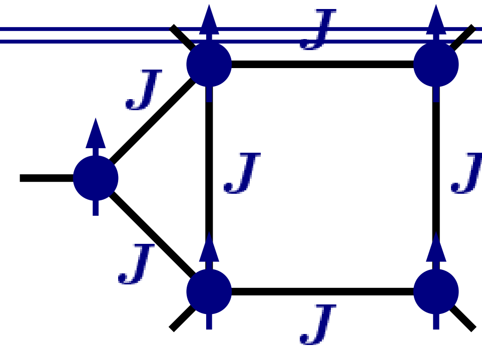

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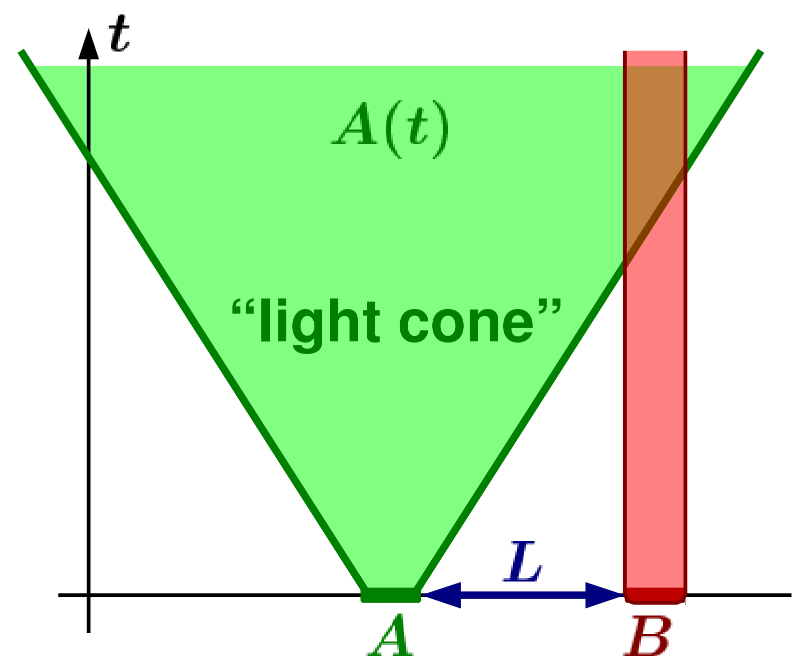

- Relevance:
  - question of fundamental interest
  - propagation speed of perturbations
  - facilitates simulation of dynamics
  - imaginary time ⇒ exponential decay of correlations

Bosonic systems

- What about **systems of interacting particles**, such as bosons?

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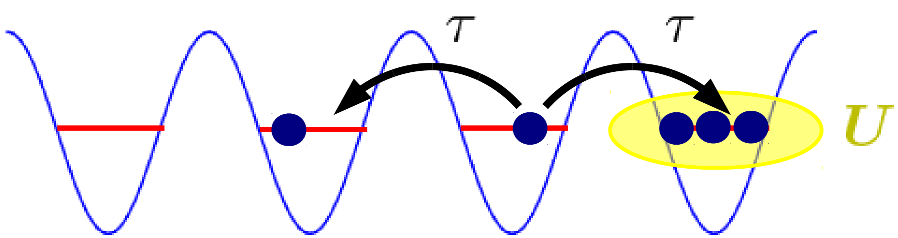

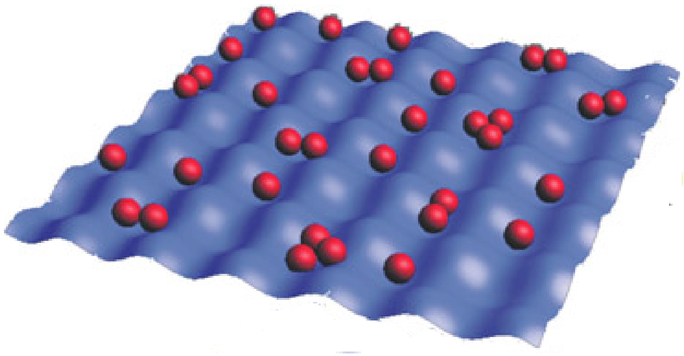

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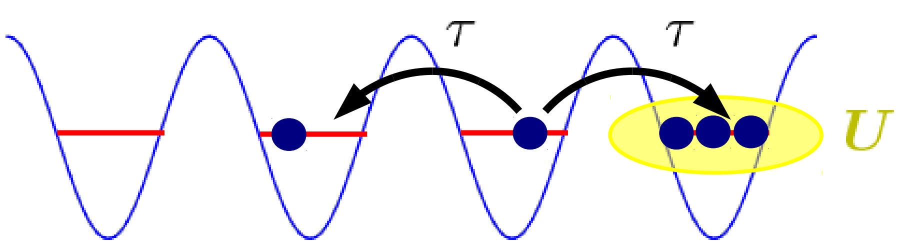

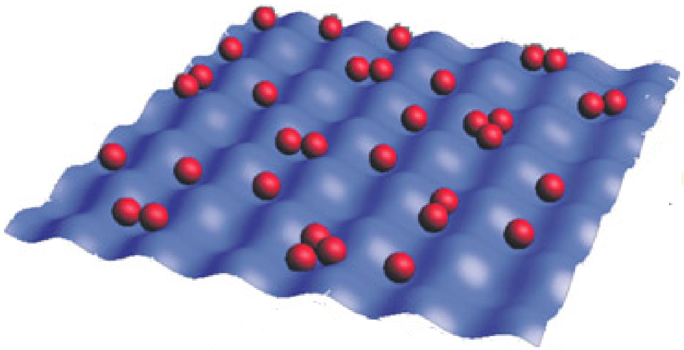

$$H_{\mathrm{BH}} = -\tau \sum_{<j,k>} (\hat{a}_j^\dagger \hat{a}_k + \hat{a}_k^\dagger \hat{a}_j) + U \sum_j \hat{n}_j(\hat{n}_j - 1)$$

$\hat{a}_j$: annihilate a particle at site j

$\hat{a}_j^\dagger$: create a particle at site j

$\hat{a}_j|n\rangle = \sqrt{n}|n-1\rangle \quad \leftrightarrow \quad \hat{n}_j = \hat{a}_j^\dagger \hat{a}_j$: counts particles at site

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$\Rightarrow$ hopping term $a_j^\dagger a_k$ unbounded (or only by $\|a_j^\dagger a_k\| \leq N_{\mathrm{tot}}$)

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- hopping rate (and thus v) will depend on the filling of the lattice:

  $\Rightarrow$ need **constraints on initial state**

- Note: bounds exist for quadr. Hamiltonians and certain perturbations thereof [Nachtergaele, Raz, Schlein, Sims 2009]

Idea: Restrict to relevant models

- Aim: propagation speed for **Bose-Hubbard type models**
- How can we obtain a meaningful propagation speed?
  - restrict to certain **initial states of interest**
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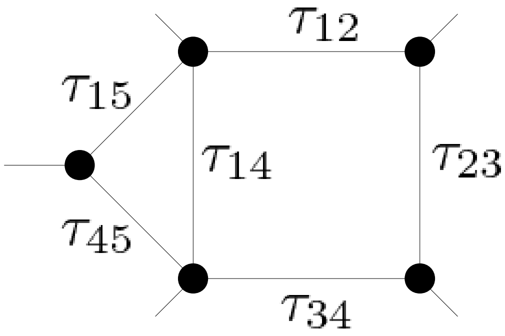
Propagation of particles in the Hubbard model



- Generalized Hubbard model:

$$H_{\mathrm{BH}} = -\sum_{<j,k>} \tau_{jk} (\hat{a}_j^\dagger \hat{a}_k + \hat{a}_k^\dagger \hat{a}_j) + f(\hat{n}_1, \ldots, \hat{n}_L)$$

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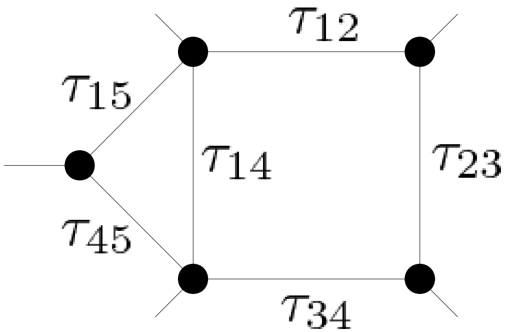

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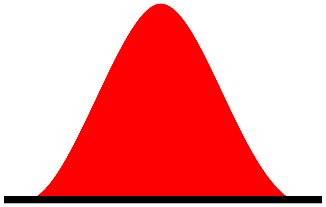
$$\boxed{\boldsymbol{\alpha_j(t) := \mathrm{tr}[\hat{n}_j \rho(t)]}}$$



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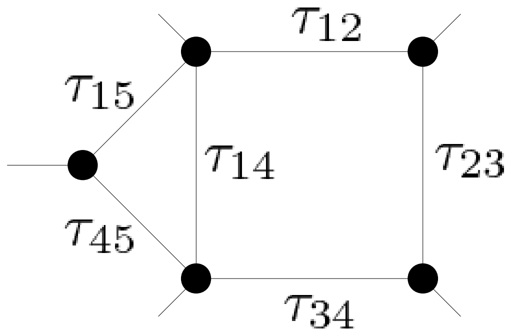

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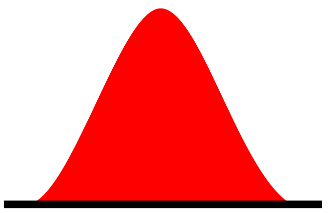


- $\dot{\rho}(t) = -i[H, \rho(t)] \;\Rightarrow\;$ differential inequality $\dot{\alpha}_j(t) \leq 2 \sum_{<j,k>} \tau_{jk} \left[\alpha_j(t)\alpha_k(t)\right]^{1/2}$

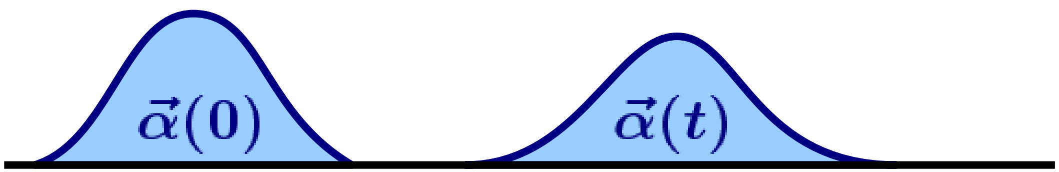
$\Rightarrow$ **worst-case upper bound $\boldsymbol{\gamma_j(t) \geq \alpha_j(t)}$** evolves according to:

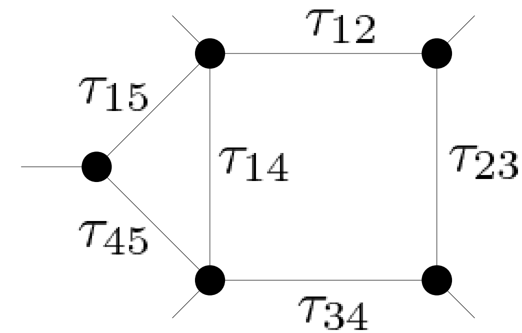
$$\dot{\gamma}_j(t) = 2 \sum_{<j,k>} \tau_{jk}(\gamma_j(t) + \gamma_k(t)) \qquad \text{(linearized)}$$

Obtaining a speed limit

- bound $\gamma_j(t) \geq \alpha_j(t)$ $\Rightarrow$ worst-case solution for propagation

$$\vec{\alpha}(t) \leq e^{Mt}\vec{\alpha}(0)$$ with M the “adjacency matrix”





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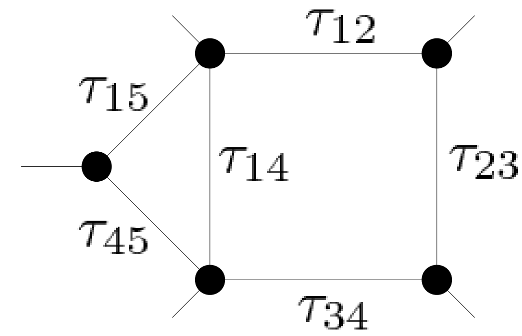

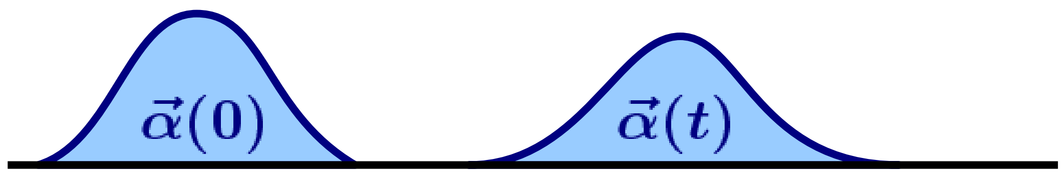

- M banded matrix $\Rightarrow$ e^{Mt} decays exponentially away from diagonal

[Benzi & Golub '99]

$$\left[e^{Mt}\right]_{jk} \leq c\, e^{vt-d(j,k)} \quad \text{with} \quad v = c_G\, \tau$$

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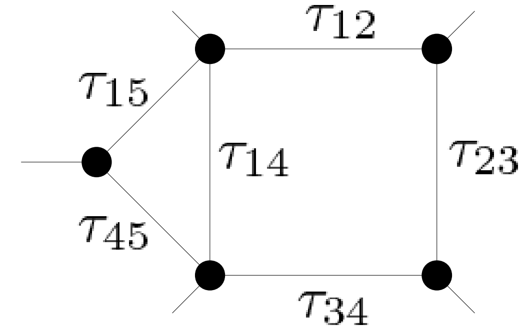

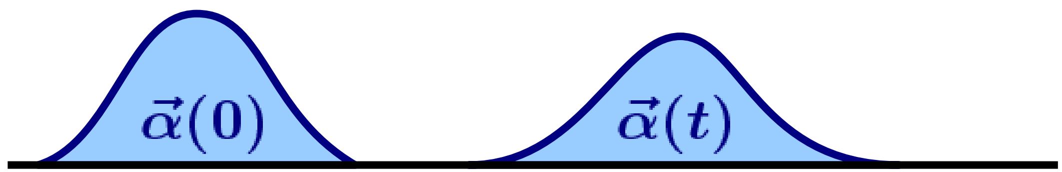

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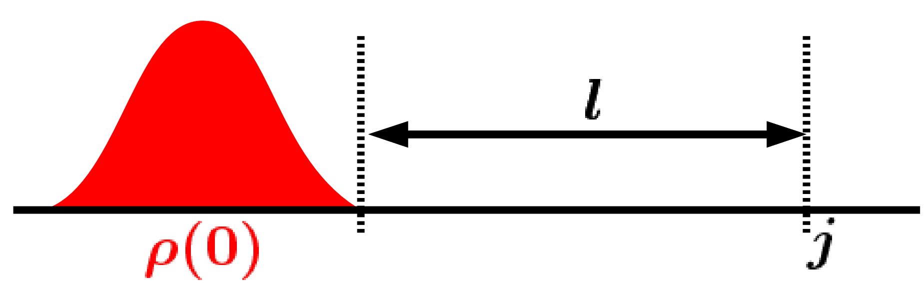
- together:

$$\boxed{\begin{gathered}\alpha_j(t) \leq C N_0 e^{vt-l} \\ v = c_G\,\tau\end{gathered}}$$

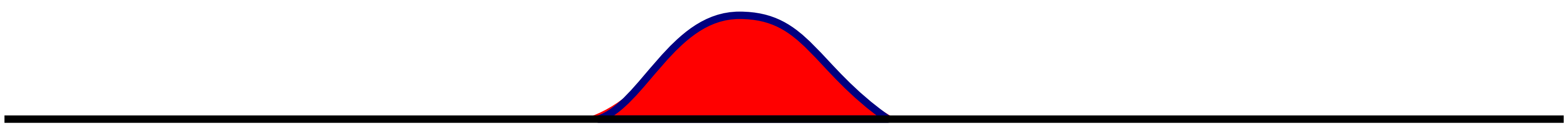


$\Rightarrow$ speed independent of particle number!

Speed limit for interacting particles

$$\alpha_j(t) \leq C N_0 e^{vt-l} \quad \textbf{where} \quad v \propto \tau$$

- Proof idea: **Study evolution of worst case bound** on $\alpha_j(t)$:



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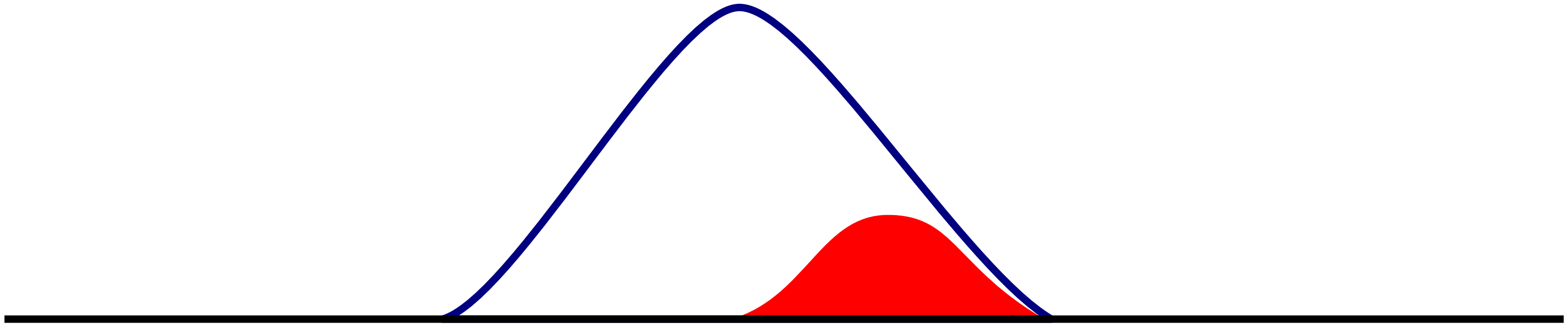

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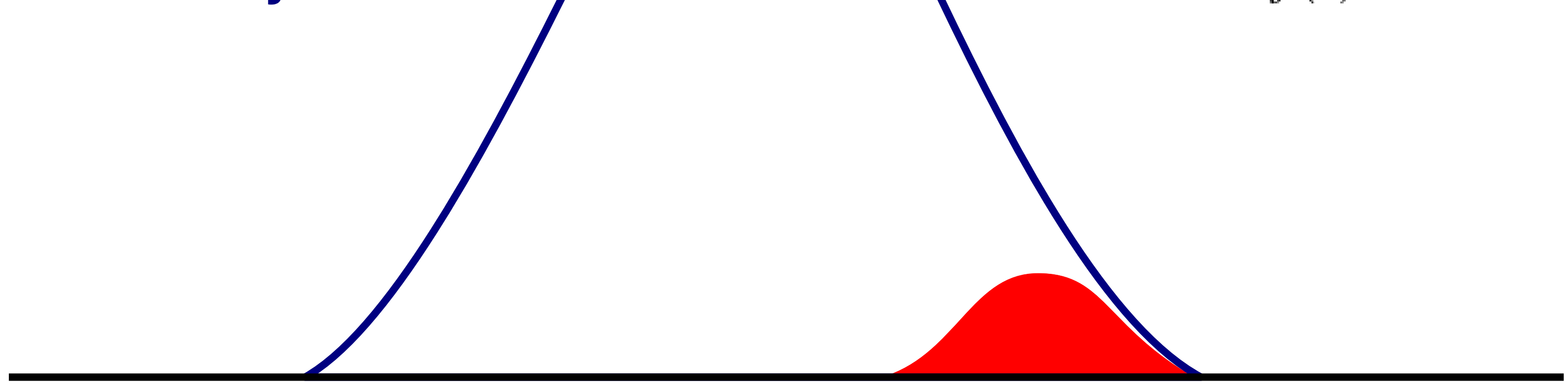

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- argument works for **any Hubbard-type model** on **any graph**
- extension possible to
  - **higher moments** of particle number
  - **arbitrary local operators**
  - **operators acting on larger blocks** (up to log-size)

… by iteratively bounding those quantities by $\alpha_j(t)$.

Extensions

- can be extended to **several species** of particles, **fermions**, **Bose-Fermi mixtures**, and even **anyons**:

$$H = -\sum_{<j,k>,s} \tau_{jk} (\hat{a}^\dagger_{j,s} \hat{a}_{k,s} + \hat{a}^\dagger_{k,s} \hat{a}_{j,s}) + f(\{n_{j,s}\}_{j,s})$$

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• works for certain **dissipative theories**, e.g. for particle losses:

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describes
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$$\Rightarrow \ \boldsymbol{\dot{\alpha}_j(t)} = \dot{\alpha}_j^{\mathrm{Ham}}(t) - \underbrace{\dot{\alpha}_j^{\mathrm{diss}}(t)}_{\geq 0} \leq \boldsymbol{\dot{\alpha}_j^{\mathrm{Ham}}(t)}$$

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- idea extendible to **continuum theories**:
  either continuum limit, or continuous differential inequalities for $\alpha(x,t)$

Summary

- We have studied the propagation of interacting bosons
- We have found a finite propagation speed for any excitation into the initially unoccupied region
- Propagation speed only depends on coupling strength
- Extends to Bose-Fermi mixtures, dissipative models, continuum theories

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