Catalysis and activation of magic states (for fault tolerance)

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Based on work from:

[Cam ‘11] arXiv:1010.0104

Builds on previous work with Dan Browne

[Cam ‘10] Phys. Rev. Lett. 104 030503 (2010)

[Cam ’09]L.N.C.S (TQC ’09) 5906 20 (2009)

Overview

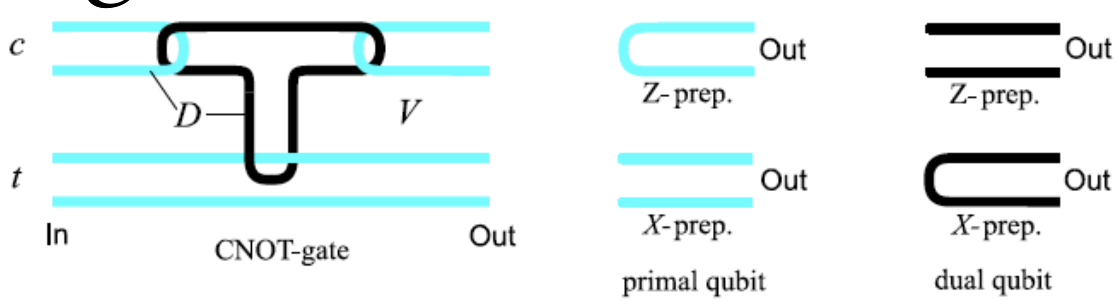
- Fault tolerance and Magic States
- Magic State Catalysis *NEW*
- Bound Magic States
- Activation (single shot and asymptotically) *NEW*

Motivations for magic states

- Magic states + Fault tolerant Clifford group = Universal Quantum computing;

e.g.1 Topological FTQC: Pfaffian states of quantum hall systems with Landau filling fraction =5

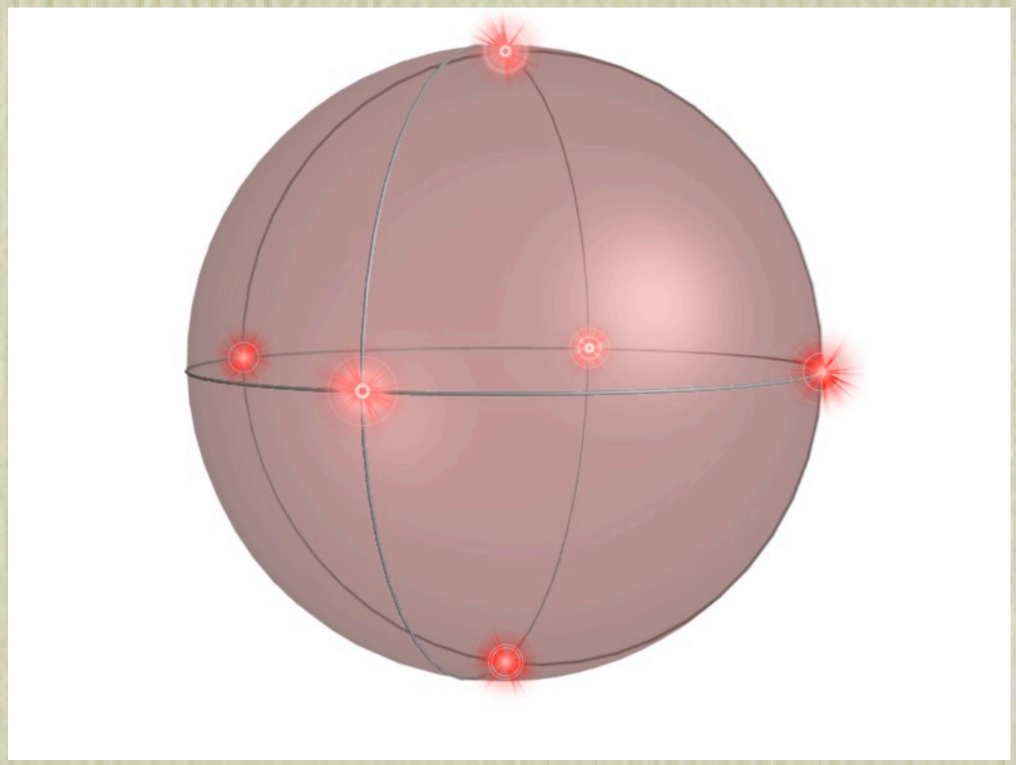
e.g. 2: Raussendorf et al

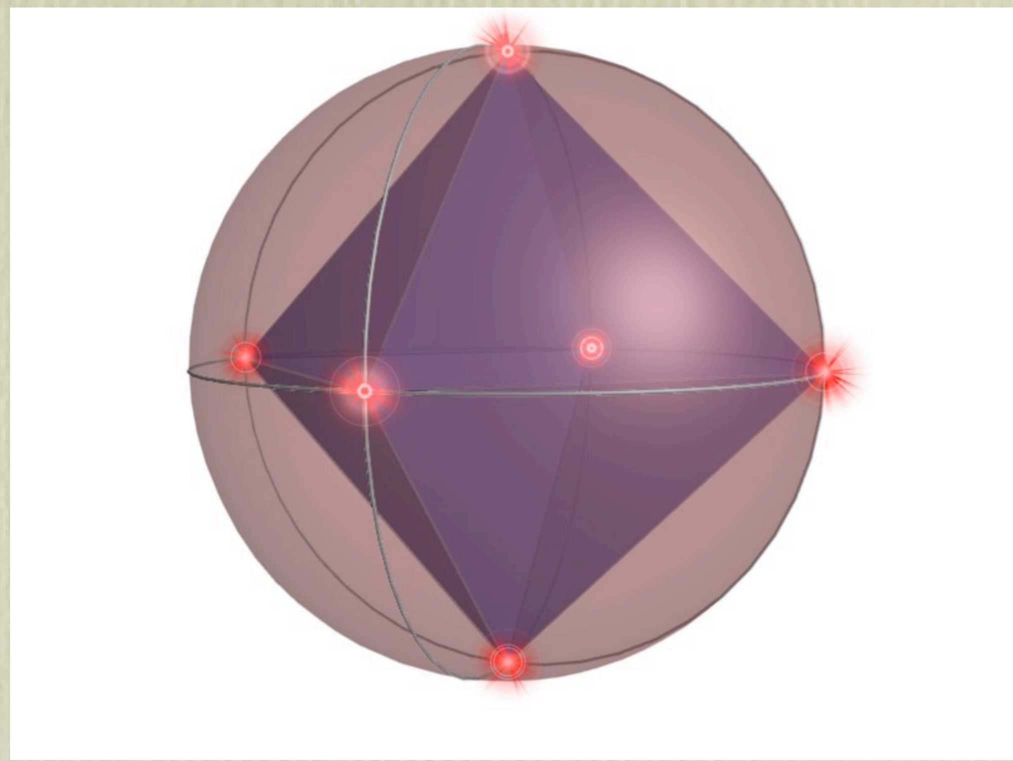


e.g. 3. most stabilizer codes, if we don't make use of Shor style methods of making Toffoli states.

- The "resource theory" of magic states shares <u>similarities with entanglement theory</u>, and this talk will explore these symmetries.

1 qubit stabilizer states





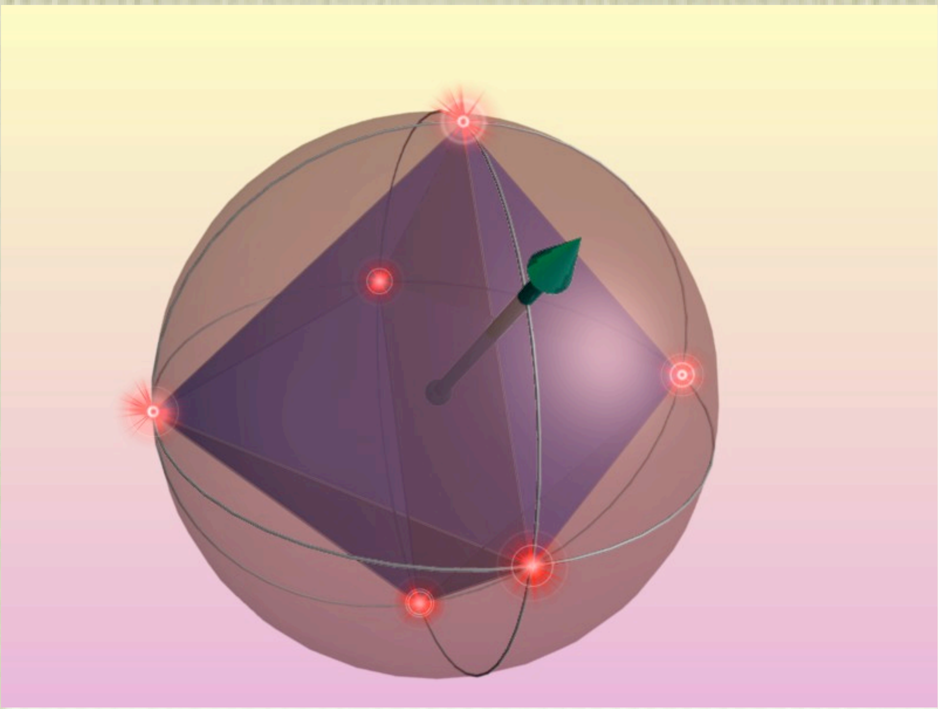
6 pure single-qubit stabilizer states.

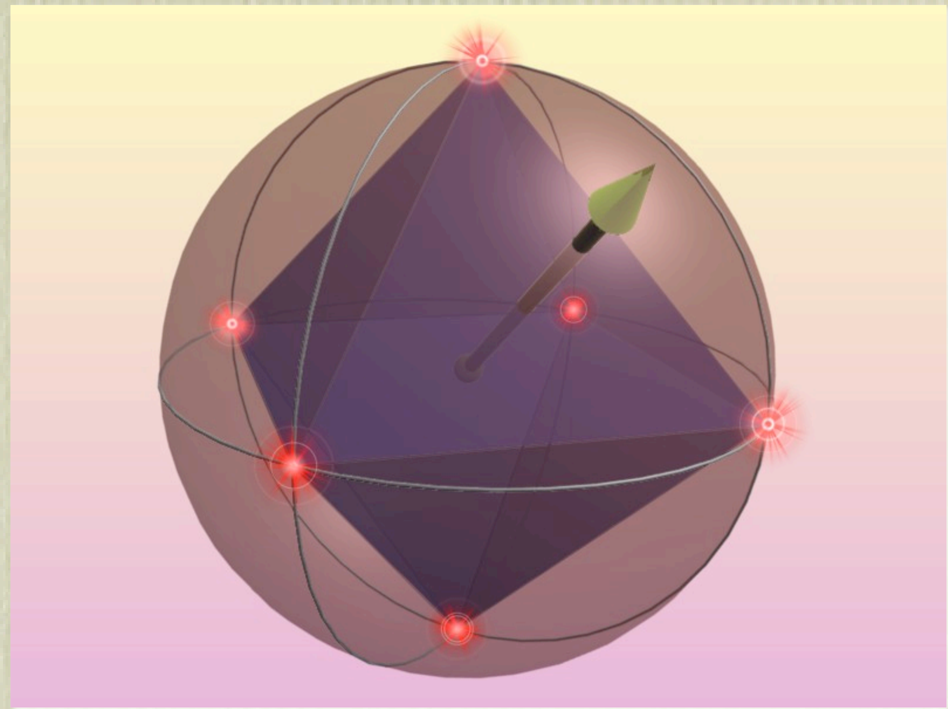
$$Z|0\rangle = |0\rangle \qquad (-Z)|1\rangle = |1\rangle$$
$$X|+\rangle = |+\rangle \qquad (-X)|-\rangle = |-\rangle$$
$$Y|\circlearrowright\rangle = |\circlearrowright\rangle \qquad (-Y)|\circlearrowleft\rangle = |\circlearrowleft\rangle$$

Mixing over these gives the ***stabilizer octahedron***.

$$\rho = \frac{1}{2}\left(1 + c_x X + c_y Y + c_z Z\right)$$
$$|c_x| + |c_y| + |c_z| \leq 1$$

Some Clifford gates





Hadamard

A 180 degree rotation about octahedron edge

$$
\begin{aligned}
HZH^\dagger &= X \\
HXH^\dagger &= Z \\
HYH^\dagger &= -Y
\end{aligned}
$$

T-Rot

A 120 degree rotation about octahedron face

$$
\begin{aligned}
TZT^\dagger &= X \\
TXT^\dagger &= Y \\
TYT^\dagger &= Z
\end{aligned}
$$

Gottesman-Knill theorem

classical computer → efficiently simulates → Circuit consisting of:

- Preparing Stabilizer States;
- Pauli measurements;
- Clifford group unitaries

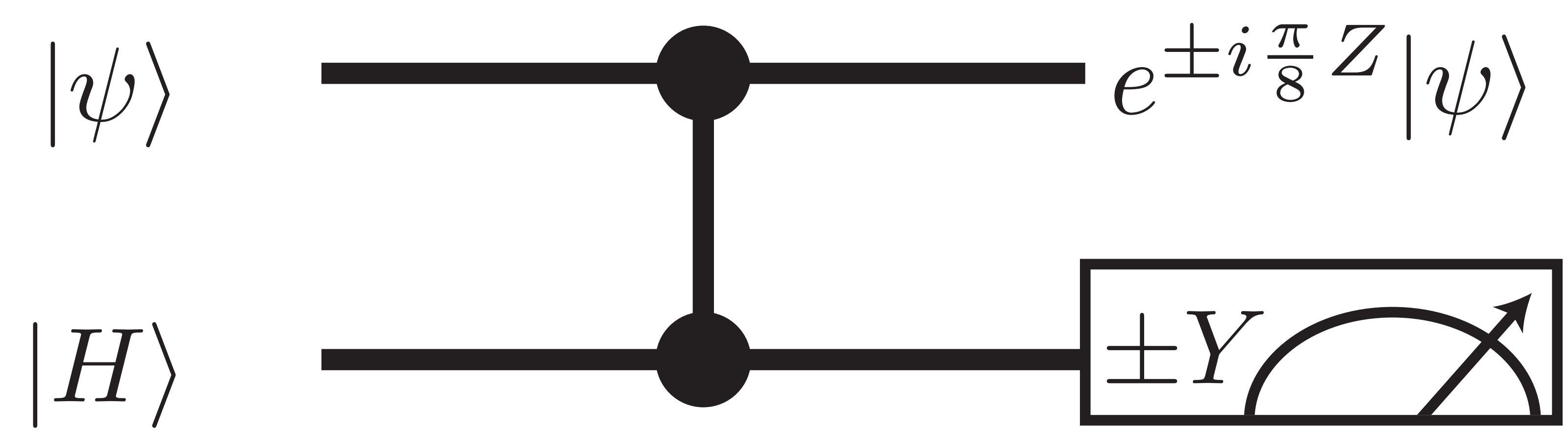
It is easy to see that n-qubit in a stabilizer state can be described by n(2n+1) bits! Also efficient in time.

[Got '98]

Promoting the Clifford group

Or a similar eigenstate on the equator....

$$H|H\rangle = |H\rangle \qquad |H\rangle\langle H| = \frac{1}{2}\left(1 + \frac{X+Z}{\sqrt{2}}\right)$$



Recap and comparison

	Magic	*Entanglement*
"Free" resource states	stabilizer states	Separable states
"Free" operations	Clifford unitaries, Pauli measurements	Local unitaries, and measurements
Ideal resource	Some pure non-stabilizer states. e.g. H state.	Pure entangled state. e.g. Bell pair

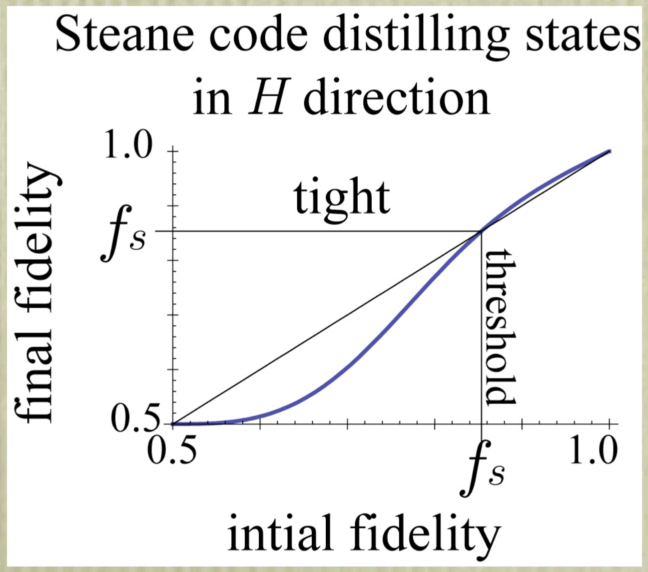
Reichardt's protocol [Rei '05, '06]

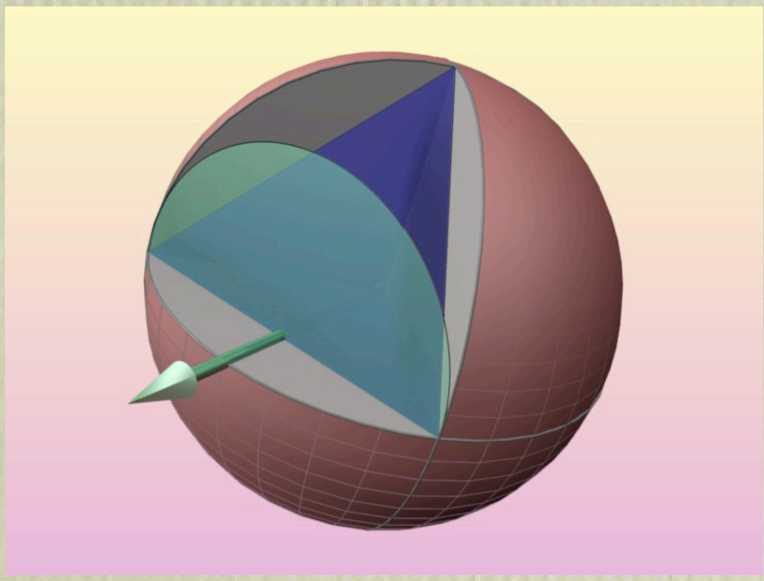
Noisy Hadamard States $\rho(f) = \frac{1}{2}\left(1 + (2f-1)\frac{X+Z}{\sqrt{2}}\right)$

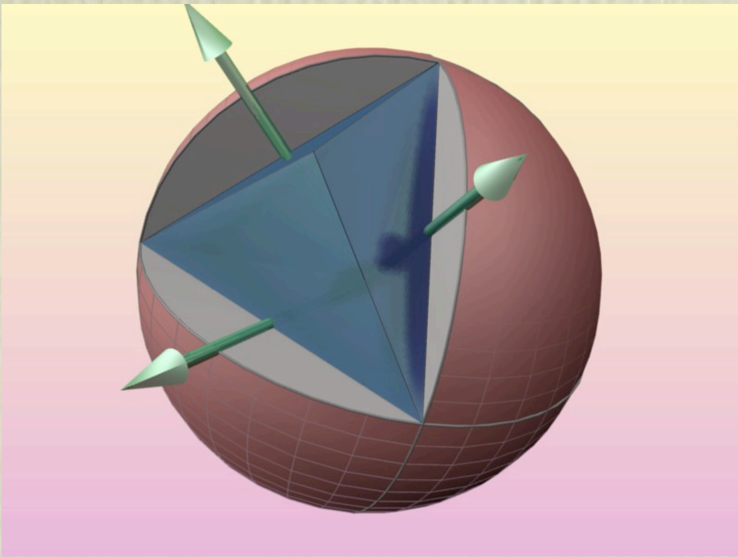
(1) Take 7 noisy H states copies and measure the 6 generators of the Steane code.

$$X_1X_2X_3X_4 1_5 1_6 1_7 \quad X_1X_2 1_3 1_4 X_5X_6 1_7 \quad X_1 1_2 X_3 1_4 X_5 1_6 X_7$$
$$Z_1Z_2Z_3Z_4 1_5 1_6 1_7 \quad Z_1Z_2 1_3 1_4 Z_5Z_6 1_7 \quad Z_1 1_2 Z_3 1_4 Z_5 1_6 Z_7$$

(2) Post-select on all syndromes "+1"& decode







Recap and comparison

	Magic	*Entanglement*
Distillation	All 1-qubit pure states, and some mixed states. [Bra '05, Rei '05, Rei '06]	All pure states and some mixed states.
Bound (undistillable) states	(will discuss later)	Yes [Horo '98]
Catalysis	?	Yes [Jon '99]
Activation	?	Yes [Horo '99]

Entanglement catalysis [Jon '99]

Banker loans client a resource (the catalyst) and demands that exactly the same state is (always) returned. The client is able to exploit the catalyst.

$$|\psi_1\rangle \not\to_D |\psi_2\rangle$$

But with catalyst

$$|\psi_1\rangle|\varphi\rangle \to_D |\psi_2\rangle|\varphi\rangle$$

$$\begin{aligned}
|\psi_1\rangle &= \sqrt{0.4}|00\rangle + \sqrt{0.4}|11\rangle + \sqrt{0.1}|22\rangle + \sqrt{0.1}\sqrt{33} \\
|\psi_2\rangle &= \sqrt{0.5}|00\rangle + \sqrt{0.25}|11\rangle + \sqrt{0.25}|22\rangle \\
|\varphi\rangle &= \sqrt{0.6}|44\rangle + \sqrt{0.4}|55\rangle
\end{aligned}$$

Magic state catalysis [Cam '11]

Banker loans client a resource (the catalyst) and demands that exactly the same state is (always) returned. The client is able to exploit the catalyst.

$$|\psi_1\rangle \nrightarrow_D |\psi_2\rangle$$

But with catalyst

$$|\psi_1\rangle|\varphi\rangle \rightarrow_D |\psi_2\rangle|\varphi\rangle$$

$$\begin{aligned} |\psi_1\rangle &= (|H_0H_0H_0\rangle + |H_1H_1H_1\rangle)/\sqrt{2} \\ |\psi_2\rangle &= |H_0\rangle \\ |\varphi\rangle &= |H_0\rangle \end{aligned}$$

$$|H_x\rangle\langle H_x| = \frac{1}{2}\left(1 + (-1)^x \frac{X+Z}{\sqrt{2}}\right)$$

Catalysis protocol

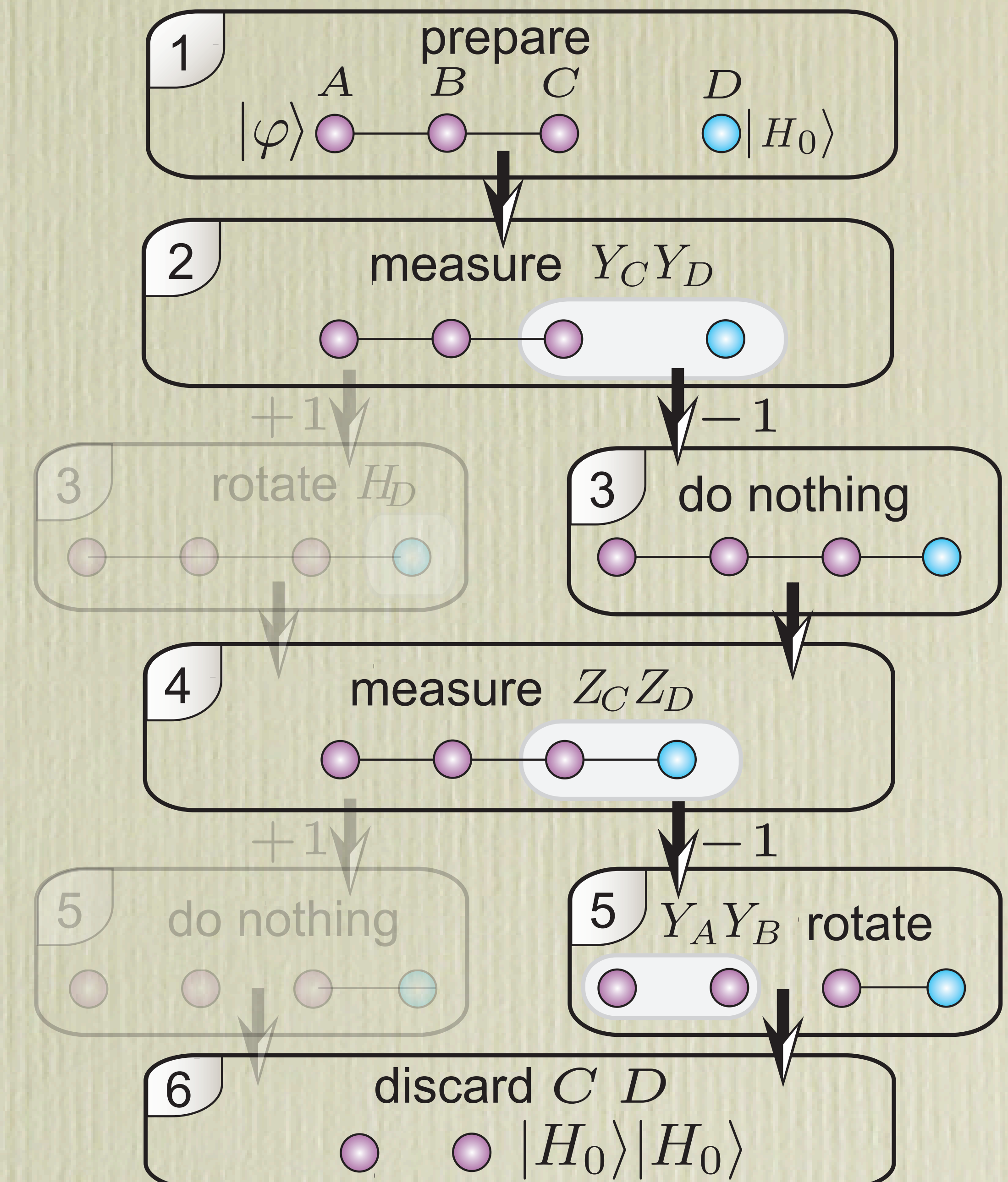
After step 4 C&D project onto the state:

$$|\Psi^-\rangle \propto |0,1\rangle - |1,0\rangle$$
$$\propto |H_0 H_1\rangle - |H_1 H_0\rangle$$

And so

$$\langle\Psi^-|_{C,D}|\psi_1\rangle_{A,B,C}|H_0\rangle_D$$
$$\propto |H_1, H_1\rangle_{A,B}$$

$$Y_A Y_B |H_1, H_1\rangle = |H_0, H_0\rangle$$

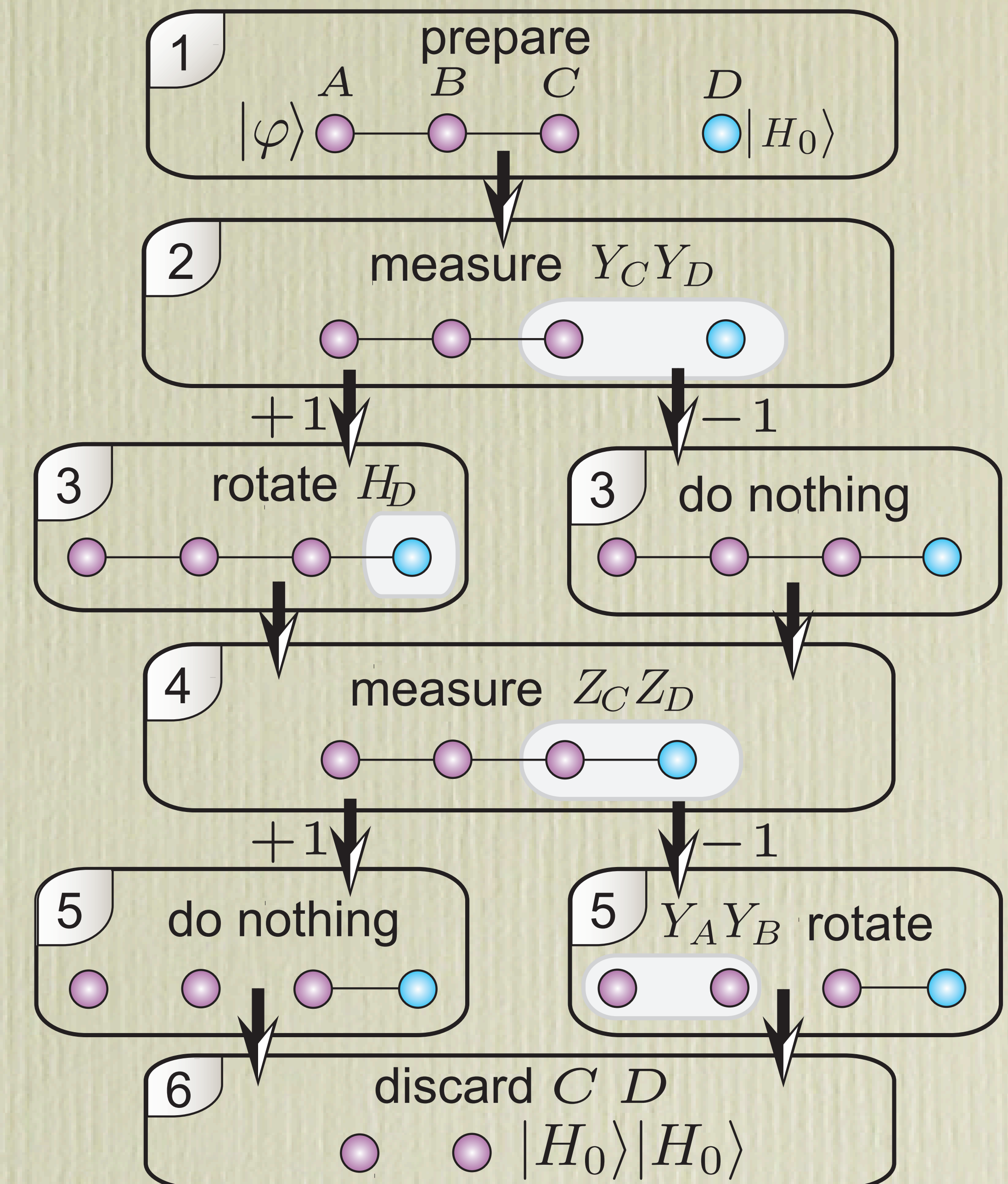


Def: $|\psi_1\rangle = (|H_0 H_0 H_0\rangle + |H_1 H_1 H_1\rangle)/\sqrt{2}$

Catalysis protocol

Steps 3 and 4 are deterministic by virtue of symmetries of |H>
e.g.

$$H_D(1+Y_CY_D)|\psi_1\rangle|H_0\rangle$$
$$=(1-Y_CY_D)H_D|\psi_1\rangle|H_0\rangle$$
$$=(1-Y_CY_D)|\psi_1\rangle|H_0\rangle$$



Def: $|\psi_1\rangle = (|H_0H_0H_0\rangle + |H_1H_1H_1\rangle)/\sqrt{2}$

Catalysis protocol

Our protocol shows

$$|H_0\rangle|\psi_1\rangle \rightarrow_D |H_0\rangle|H_0\rangle$$

To demonstrate Catalysis we require also

$$|\psi_1\rangle \nrightarrow_D |H_0\rangle$$

We prove the stronger result that

$$|\psi_1\rangle \nrightarrow_P |H_0\rangle$$

we use P to denote probabilistic transforms

Proof Outline: The ratios of the computational amplitudes for the Hadamard state are irrational. The transformations possible only give rational ratios.

Recap and comparison

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Bound (undistillable) states	(will discuss later)	Yes [Horo '98]
Catalysis	Yes. At least for Hadamard states! [Cam '11]	Yes [Jon '99]
Activation	?	Yes [Horo '99]

Bound entanglement [Horo '98]

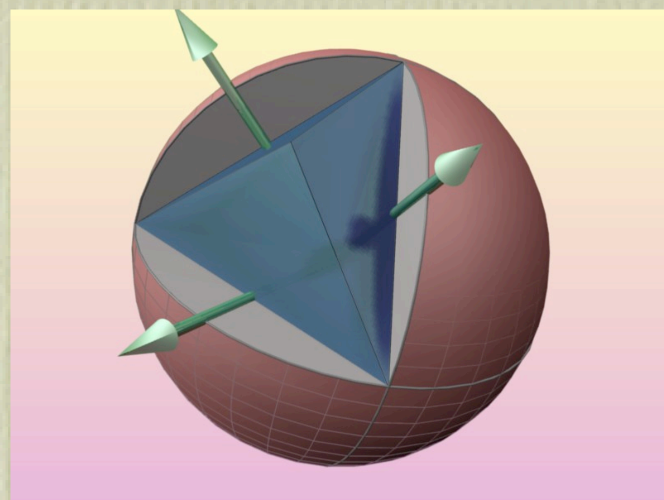
Reducibility: $\exists \sigma \in \mathcal{E}_{2\times 2}, \rho \rightarrow_P \sigma$
Irreducibility: $\nexists \sigma \in \mathcal{E}_{2\times 2}, \rho \rightarrow_P \sigma$

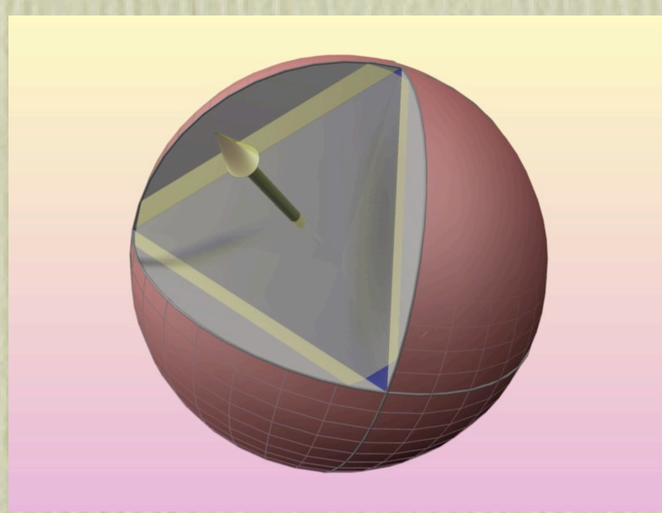
$\mathcal{E}_{2\times 2}$ Set of 2 qubit entangled states

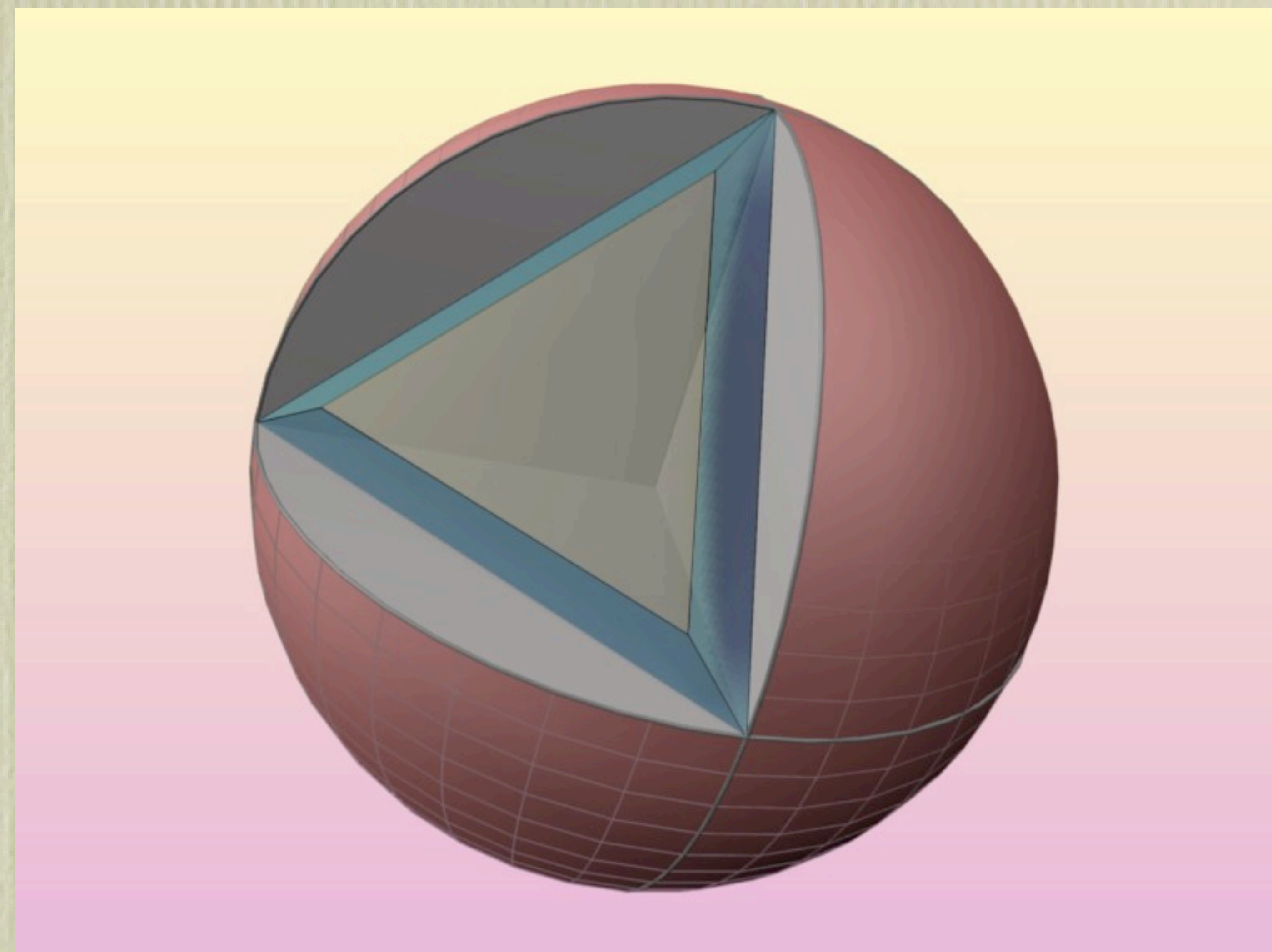
All reducible states are distillable

If the state is PPT, such that $\mathrm{tr}(|\rho^{T_B}|) = 1$ then $\forall n, \rho^{\otimes n}$ is irreducible, and so we say ρ is bound entangled

The distillable region [Bra ’05, Rei ’05, Rei ‘06]





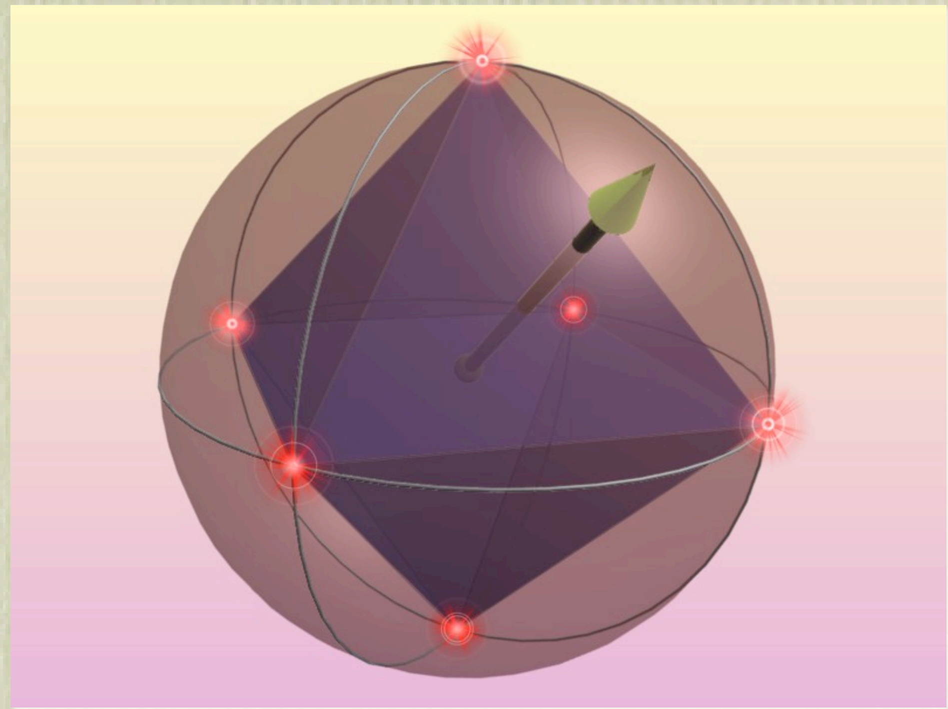


Reichardt has further increased this region. However, the reduced region is still ***not tight, except at the octahedron edges.***

T Magic states

$$T|T_{0,1}\rangle \propto |T_{0,1}\rangle$$

$$\begin{aligned}\tau(f) &= f|T_0\rangle\langle T_0| + (1-f)|T_1\rangle\langle T_1| \\ &= \tfrac{1}{2}\left(1 + (2f-1)\tfrac{X+Y+Z}{\sqrt{3}}\right)\end{aligned}$$



In the range:

$$\frac{1}{2}\left(1+\frac{1}{\sqrt{3}}\right) < f \leq \frac{1}{2}\left(1+\sqrt{\frac{3}{7}}\right)$$

We have non-stabilizer states that cannot be distilled by any known protocol

T-Rot

A 120 degree rotation about octahedron face

$$Z \to X$$
$$X \to Y$$
$$Y \to Z$$

Boundness of T-magic states

Previous results* tell us that:

Theorem 2 *For any finite n, there exists a positive $\epsilon_n > 0$, and a corresponding no-go region of fidelities $f \leq f_{\mathrm{st}} + \epsilon_n$. Inside this no-go region, it follows that for any single qubit state, ρ, we have that $\tau(f)^{\otimes n} \to_P \rho$ if and only if $\tau(f)^{\otimes n} \to_P \rho$. We say that the family of states $\tau(f)$ is bound.*

Roughly, there exist bound Magic states when we have only a finite number of copies

* : Phys. Rev. Lett *104* 030503 (2010)

Entanglement Activation [Horo '99]

Consider a bound state ρ which we suspect (pre 1999) is not useful for any task!

For some such states there exist activators σ_{ACT} such that

$$\sigma_{ACT} \otimes \rho^{\otimes n} \rightarrow_P \sigma \qquad \lim_{n\rightarrow\infty} \sigma \rightarrow |\Psi^+\rangle\langle\Psi^+|$$

However $\sigma_{ACT} \nrightarrow_P \sigma$

Bell pair

Single-shot activation

Theorem 3 *Magic activation is possible: For the activator* $\sigma(q) = q\tau_{0,1} + (1-q)\tau_{1,0}$ *(for some* $1 > q > 1/2$*) and any* $\tau(f)$*, with* $f_{\mathrm{st}} < f$*, there exist a single-qubit state* ρ *such that:*

i. $\sigma(q) \otimes \tau(f) \to_P \rho$*; even though*

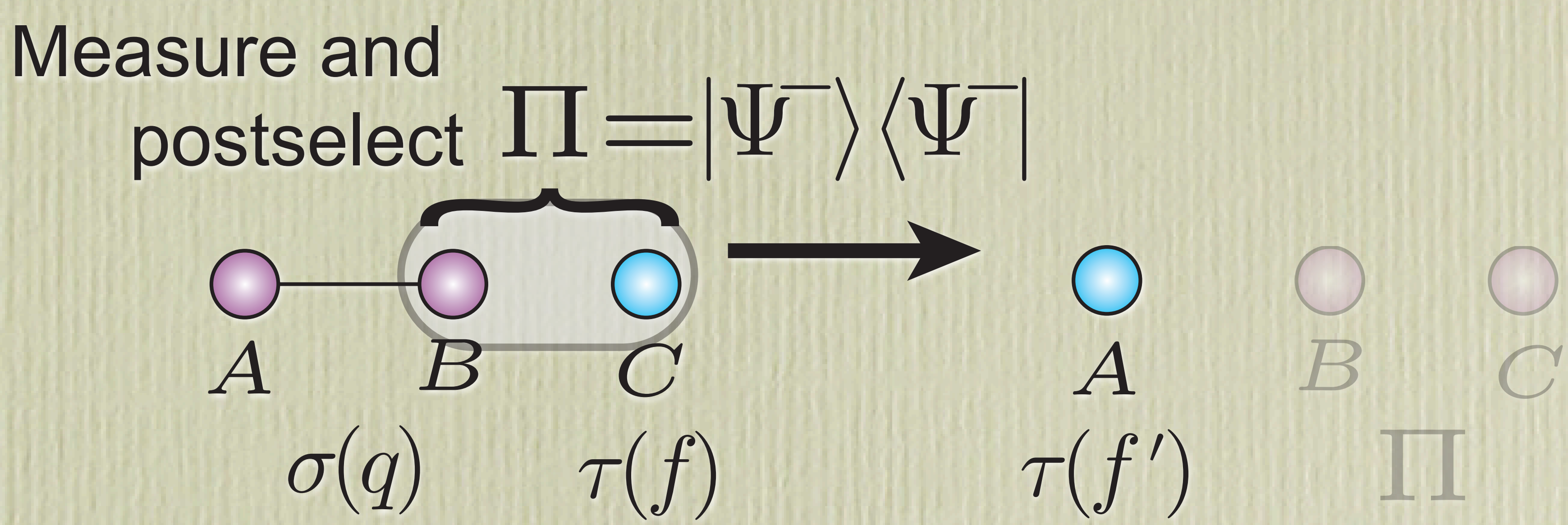
ii. $\sigma(q) \nrightarrow_P \rho$*; and*

iii. $\tau(f) \nrightarrow_P \rho$*.*

$$\sigma(q) \quad = \quad q\tau_{0,1} + (1-q)\tau_{1,0}$$

$$\tau_{x,y} = |T_x, T_y\rangle\langle T_x, T_y|$$

single-shot activation



So... $\sigma(q) \otimes \tau(f) \rightarrow_P \tau(f')$

where $f' = \dfrac{qf}{qf + (1-q)(1-f)}$

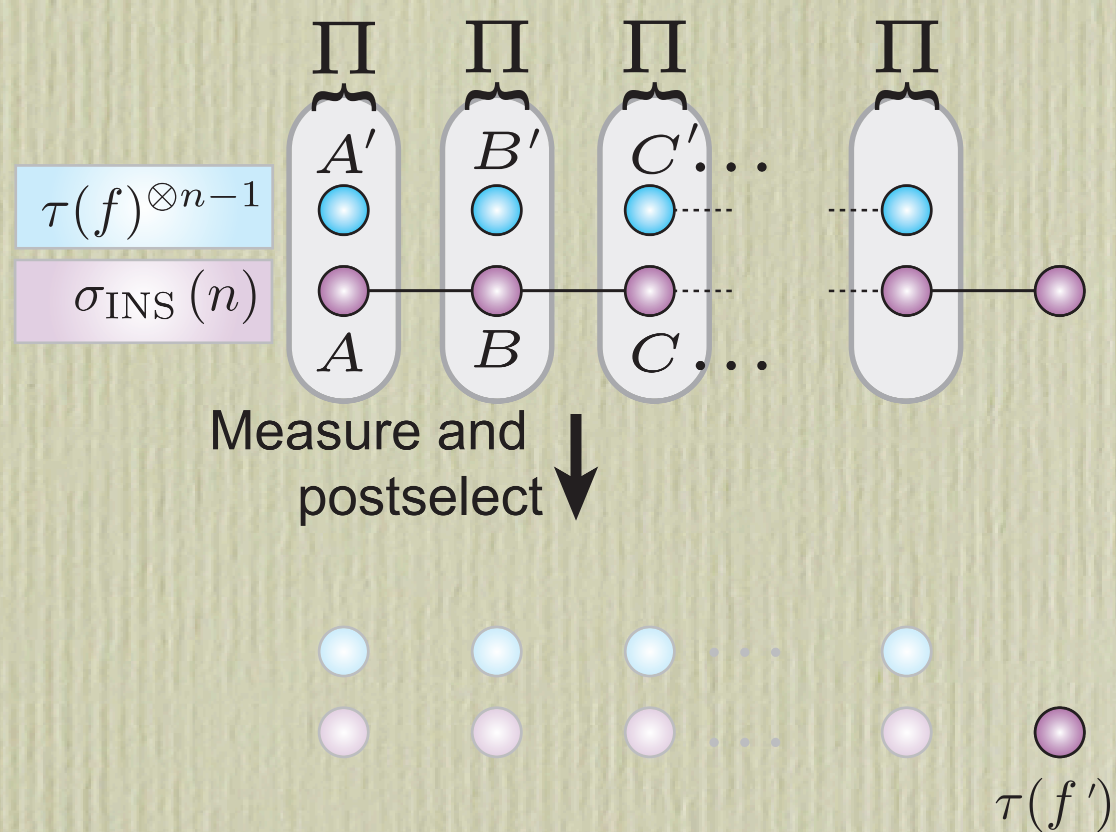
One can also verify that

$\sigma(q) \nrightarrow \tau(f')$

$$\begin{aligned} |\Psi^-\rangle &= (|0,1\rangle - |1,0\rangle)/\sqrt{2} \\ &\propto |T_0, T_1\rangle - |T_1, T_0\rangle \end{aligned}$$

$$\begin{aligned} \sigma(q) &= q\tau_{0,1} + (1-q)\tau_{1,0} \\ \tau(f) &= f\tau_0 + (1-f)\tau_1 \end{aligned}$$

Asymptotic activation

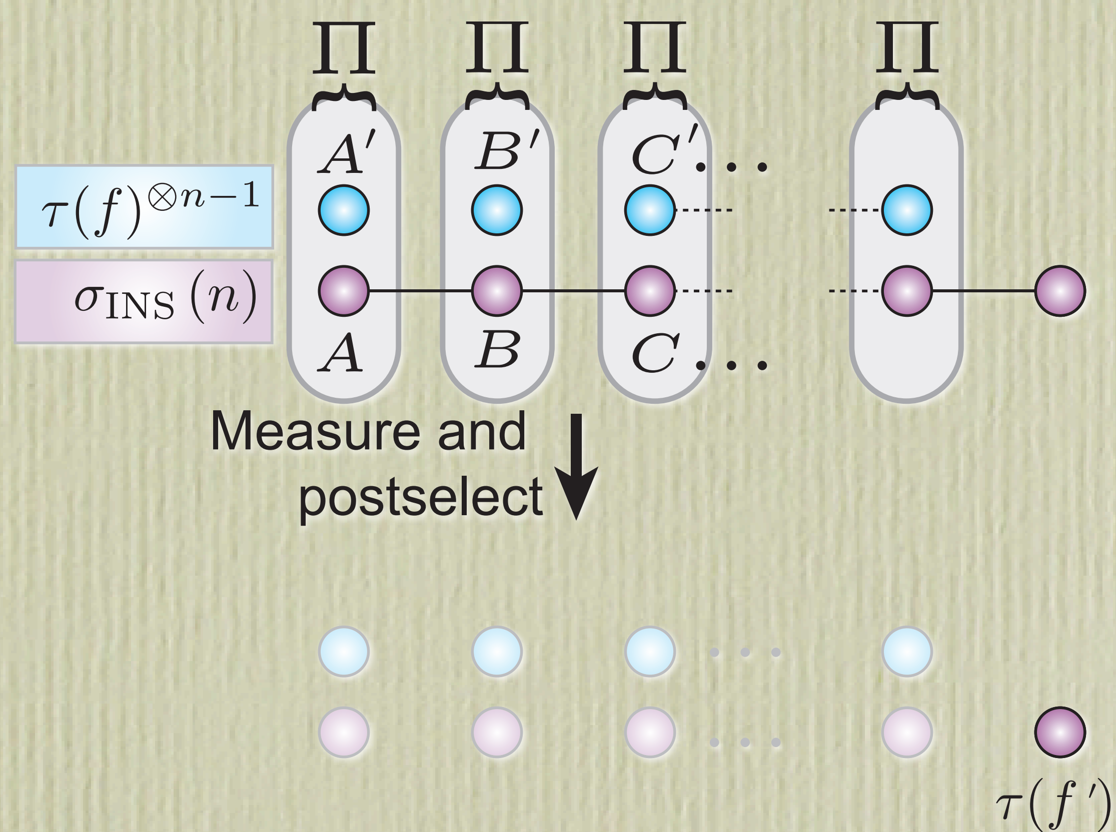


Using the same techniques as before we can verify that

$$\sigma_{\mathrm{INS}}(n) \otimes \tau(f)^{n-1} \rightarrow \tau(f')$$

with $\lim_{n\rightarrow\infty} f' = 1$

Asymptotic activation



Using the same techniques as before we can verify that

$$\sigma_{\text{INS}}(n) \otimes \tau(f)^{n-1} \to \tau(f')$$

with $\lim_{n\to\infty} f' = 1$

$$\sigma_{\text{INS}}(n) \quad = \quad q_n \tau_0^{\otimes n} + (1-q)\frac{1}{2^n}$$

Irreducible Non-stabilizer State [Rei '06]

$$\sigma_{\text{INS}}(n) \not\to_P \rho_{\text{nonstab}}$$

Open question:
are one-copy irreducible states, also many copy irreducible?

Recap and comparison

	Magic	*Entanglement*
Distillation	All 1-qubit pure states, and some mixed states.	All pure states and some mixed states.
Bound (undistillable) states	Yes. At least in finite regime! [Cam '09 '10]	Yes [Horo '98]
Catalysis	Yes. At least for Hadamard states! [Cam '11]	Yes [Jon '99]
Activation	Yes for single shot. Yes asymptotically using a growing resource! [Cam '11]	Yes [Horo '99]

Thanks to Dan Browne, Anwar, Matty Hoban for useful discussions.

References:

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[How '10] Howard, Vam Dam, arXiv:1011.2497
[Rat '10] Ratanje, Virmani, arXiv:1007.3455

Catalysis No-Go details

$$|\varphi\rangle = (|H_0 H_0 H_0\rangle + |H_1 H_1 H_1\rangle)/\sqrt{2}$$

is Clifford equivalent to

$$\begin{aligned} |\varphi'\rangle &= (|H'_0 H'_0 H'_0\rangle + |H'_1 H'_1 H'_1\rangle)/\sqrt{2} \\ &= (|0,0,0\rangle + i|1,1,0\rangle + i|1,0,1\rangle + i|0,1,1\rangle)/2 \end{aligned}$$

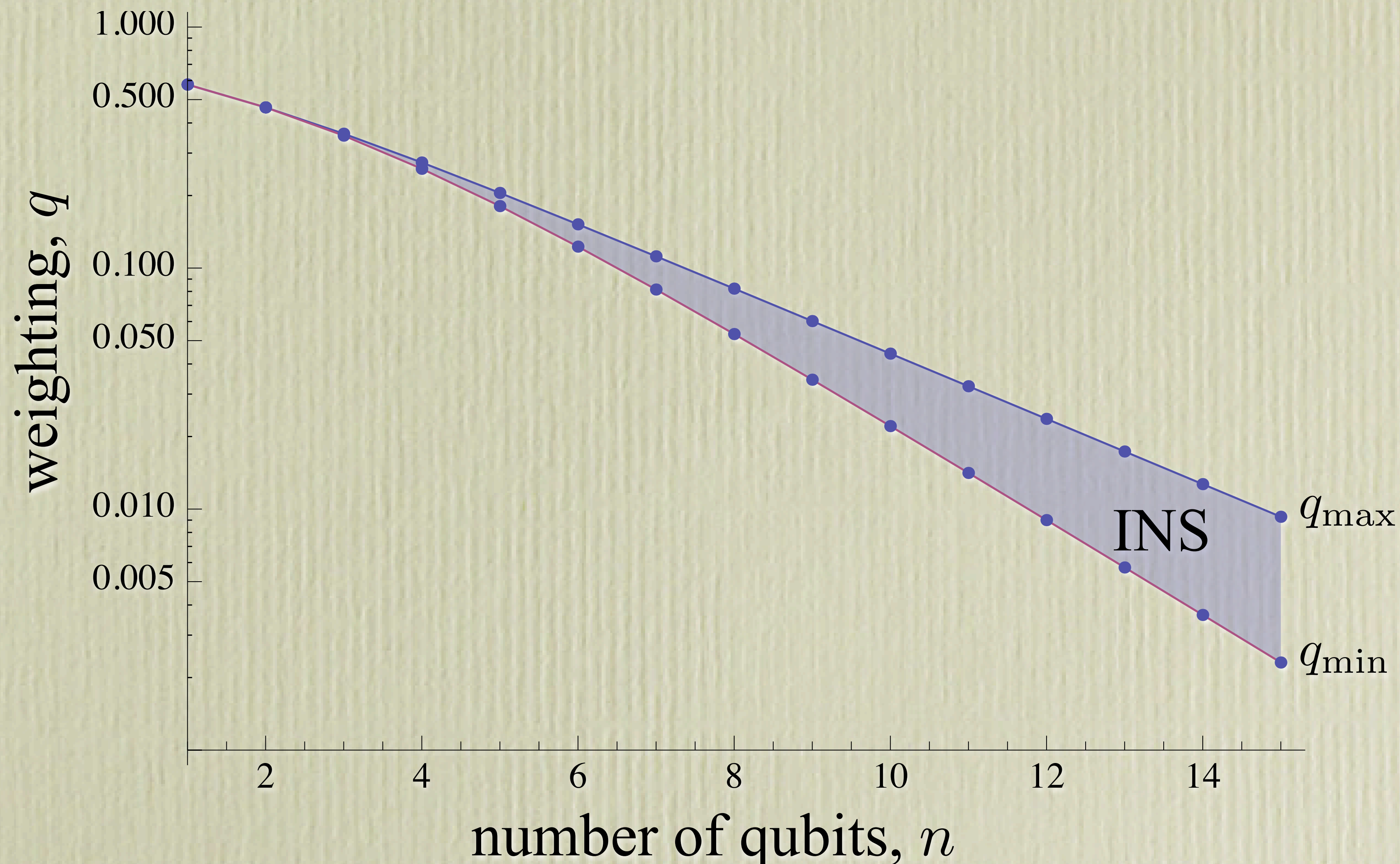
All pure transforms to one-qubit have the form

$$|\varphi'\rangle \rightarrow \langle 0_L|\varphi'\rangle|0\rangle + \langle 1_L|\varphi'\rangle|1\rangle$$

Define: $R = \dfrac{|\langle 0_L|\varphi'\rangle|^2}{|\langle 1_L|\varphi'\rangle|^2}$ Target is: $\dfrac{|\langle 0_L|H_0\rangle|^2}{|\langle 1_L|H_0\rangle|^2} = 3 - 2\sqrt{2}$

However R is rational $R = \dfrac{a^2+b^2}{c^2+d^2}$ $\begin{aligned} a,b,c,d &\in \mathbb{Z} \\ |a|,|b|,|c|,|d| &\leq 4 \end{aligned}$

Irreducible NS



$$q_{\max} = [1 + (2f_{\mathrm{st}})^{n-1}(\sqrt{3} - 1)]^{-1},$$

$$q_{\min} = (2^n - 1)/[(1 + \sqrt{3})^n - 1].$$

Irreducible NS

- To find q_{min} we must verify that the state is indeed a non-stabilizer state. We do this by calculating ||rho||_{st}

Lemma 1 *A density matrix ρ, with decomposition in the Pauli basis $\rho = \sum_j a_j \sigma_j$, is a nonstabilizer state if*

$$||\rho||_{\mathrm{st}} = \sum_j |a_j| > 1. \tag{19}$$

Irreducible NS

$$\rho_{\text{out}} = \frac{q.\Pi\tau_0^{\otimes n}\Pi + (1-q)\Pi/2^n}{q.\text{tr}(\Pi\tau_0^{\otimes n}) + (1-q)/2^{n-1}}. \qquad (22)$$

The largest eigenvalue of the projected state is

$$\lambda = \frac{q.\text{tr}(\Pi\tau_0^{\otimes n}) + (1-q)/2^n}{q.\text{tr}(\Pi\tau_0^{\otimes n}) + (1-q)/2^{n-1}}. \qquad (23)$$

To make further progress we must evaluate the maximum possible value of $\text{tr}(\Pi\tau_0^{\otimes n})$.

Lemma 2 *For n copies of a single-qubit state, τ_0, and for all projectors, Π, onto a 2^m-dimensional stabilizer subspace, the maximum probability of projection is*

$$\max_{\Pi}\left[\text{tr}(\Pi\tau_0^{\otimes n})\right] = f_{\text{st}}^{n-m} \qquad (24)$$