Tsirelson's problem and Kirchberg's conjecture

arXiv:1008.1168v2

Tobias Fritz

Institute of Photonic Sciences (ICFO), Barcelona, Spain

14th Workshop on Quantum Information Processing
Singapore, 11 Jan 2011

Overview

- Quantum-mechanical axioms of composite systems.
- Their significance for nonlocality theory. (Tsirelson's problem)
- Strong ties to an open problem on C^*-algebras. (Kirchberg's conjecture)
- Two new paradigms of quantum correlations.

Disclaimer: Some of these results were obtained independently in

Junge, Navascués, Palazuelos, Pérez-García, Scholz, Werner,
Connes' embedding problem and Tsirelson's problem,
J. Math. Phys. 52, 012102 (2011).

Composite systems in quantum theory I

Tensor product assumption:

- The state space of a joint system composed out of two subsystems is a tensor product

$$\mathcal{H}_A \otimes \mathcal{H}_B$$

  with local observables

$$A \otimes \mathbb{1} \text{ for } A \in \mathcal{B}(\mathcal{H}_A), \quad \mathbb{1} \otimes B \text{ for } B \in \mathcal{B}(\mathcal{H}_B)\,.$$

- Posited by standard quantum theory.
- The composite system can be constructed from the subsystems.

Composite systems in quantum theory II

Commutativity assumption:

- State space of a joint system is a Hilbert space $\mathcal{H}$ with local observables

$$A, B \in \mathcal{B}(\mathcal{H}) \ \text{such that} \ AB = BA\,.$$

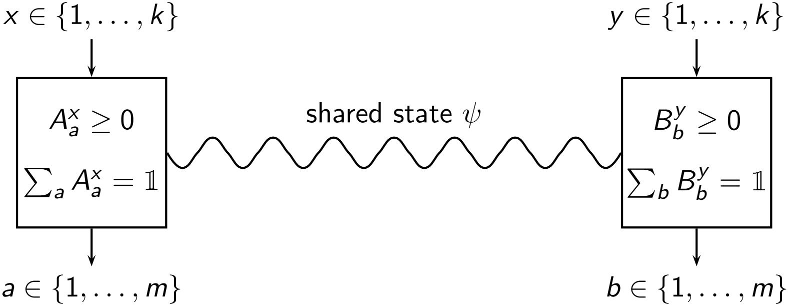
- The joint state space is in general not uniquely determined by the subsystems.
- Defining $\mathcal{H} \equiv \mathcal{H}_A \otimes \mathcal{H}_B$ obviously works
  $\Rightarrow$ tensor product assumption is a special case.
- For finite-dimensional systems, it is essentially equivalent to the tensor product assumption.
- Not so in infinite dimensions!
  ($\rightarrow$ This is what the theory of C^*-tensor products is about.)

Composite systems in quantum theory III

Philosophical observation:

- Nature does not construct composite systems from subsystems.
- Rather, she presents us composite systems which we perceive as made out of subsystems.
- $\Rightarrow$ Correct question: when does a system look like it were composed out of subsystems?
- One possible answer: the commutativity assumption.

Nonlocality theory: Bell scenarios

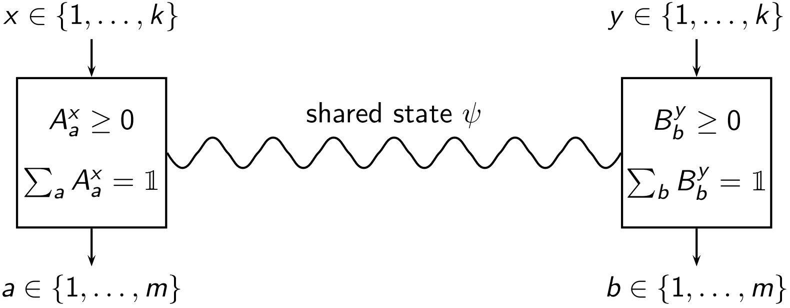


- with the tensor product assumption:

$$A_a^x \in \mathcal{B}(\mathcal{H}_A)\,, \qquad B_b^y \in \mathcal{B}(\mathcal{H}_B)\,, \qquad \psi \in \mathcal{H}_A \otimes \mathcal{H}_B$$

$$\boxed{P(a, b|x, y) = \langle \psi, (A_a^x \otimes B_b^y)\, \psi \rangle}$$

Nonlocality theory: Bell scenarios



- with the commutativity assumption:

$$A_a^x, B_b^y \in \mathcal{B}(\mathcal{H}), \qquad A_a^x B_b^y = B_b^y A_a^x, \qquad \psi \in \mathcal{H}$$

$$\boxed{P(a, b|x, y) = \langle \psi, A_a^x B_b^y \psi \rangle}$$

Sets of quantum correlations

- Two sets of quantum correlations $P(a, b|x, y) \in \mathbb{R}^{k^2 m^2}$,
  1. With the tensor product assumption: $\mathcal{Q}_\otimes$.
  2. With the commutativity assumption: $\mathcal{Q}_c$.
- $\mathcal{Q}_\otimes$ is convex, $\mathcal{Q}_c$ is closed convex.
- It is unclear whether $\mathcal{Q}_\otimes$ is closed; consider the closure $\overline{\mathcal{Q}}_\otimes$.
- Tensor product assumption is a special case of the commutativity assumption $\Rightarrow \overline{\mathcal{Q}}_\otimes \subseteq \mathcal{Q}_c$.
- **Tsirelson's problem:** $\boxed{\overline{\mathcal{Q}}_\otimes \stackrel{?}{=} \mathcal{Q}_c}$

More on Tsirelson's problem

Tsirelson's problem: $\boxed{\overline{\mathcal{Q}_\otimes} \stackrel{?}{=} \mathcal{Q}_c}$

- The answer may depend on the Bell scenario considered.
- Every $P(a,b|x,y) \in \mathcal{Q}_c$ coming from a state with finite-dimensional $\mathcal{H}$ lies also in $\mathcal{Q}_\otimes$.
- No further results are known.
- Physical relevance of a potential negative answer is unclear.

Additional motivation from nonlocality theory:

- Most (all?) examples of quantum correlations use the tensor product assumption.
- Most (all?) upper bounds on quantum correlations use the commutativity assumption. (E.g. the semidefinite hierarchy.)

$\rightarrow$ Tsirelson's problem: will improvement of lower and upper bounds lead to convergence, or will there remain a gap?

Kirchberg's QWEP conjecture

Let $\mathbb{F}_2$ be the free group on two generators. For a discrete group G, $C^*(G)$ denotes its maximal group C^*-algebra.

- **Kirchberg's QWEP conjecture:**

$$\boxed{C^*(\mathbb{F}_2) \otimes_{\max} C^*(\mathbb{F}_2) \stackrel{?}{=} C^*(\mathbb{F}_2) \otimes_{\min} C^*(\mathbb{F}_2)}$$

- Proposed by Kirchberg in 1993[1].
- Equivalent to Connes' embedding problem from 1976.
- Many reformulations exist as questions on C^*-algebras, von Neumann algebras, operator spaces...

[1] E. Kirchberg, *On nonsemisplit extensions, tensor products and exactness of group C^*-algebras*, Invent. Math. (1993).

From Kirchberg's conjecture to Tsirelson's problem

Theorem

If QWEP is true, then $\overline{\mathcal{Q}}_{\otimes} = \mathcal{Q}_c$ in all Bell scenarios.

The proof relies on a characterization of both $\overline{\mathcal{Q}}_{\otimes}$ and $\mathcal{Q}_c$ in terms of group C^*-algebras.

Quantum correlations and group C^*-algebras

Sketch of the connection to group C^*-algebras:

- Label the outcomes of an m-outcome measurement by the mth roots of unity $e^{2\pi i \frac{j}{m}}$.
  $\Rightarrow$ Observable is a unitary operator of order m.
- k m-outcome measurements correspond to k unitaries of order m.
- This is equivalent to a unitary representation of the discrete group

$$\Gamma = \underbrace{\mathbb{Z}_m * \ldots \mathbb{Z}_m}_{k \text{ factors}},$$

  and hence to a representation of $C^*(\Gamma)$.
- This allows the use of methods from C^*-algebra theory, group theory, and representation theory.
- Example application: CHSH is simple because $\mathbb{Z}_2 * \mathbb{Z}_2 \cong \mathbb{Z} \rtimes \mathbb{Z}_2$.

Spatiotemporal correlations I

- In a usual Bell scenario, each party conducts exactly one measurement per run.
- More generally, one can consider scenarios where each party is allowed to conduct several measurements per run in temporal succession. $\rightarrow$ "Spatiotemporal correlations"
- Motivated by the group C^*-algebra approach to quantum correlations.
- There are examples of A_a^x and B_B^y and an initial state ψ such that the ensuing spatial correlations are local, but the spatiotemporal correlations prove nonlocality.

Spatiotemporal correlations II

Tsirelson's problem can be formulated analogously for spatiotemporal correlations.

Theorem

If QWEP is true, then Tsirelson's problem for spatiotemporal correlations also has a positive answer.

The proof is exactly analogous to the purely spatial case.

Steering data I

- If Alice conducts measurement x and gets the outcome a, then Bob's system collapses to the state

$$\rho(a|x) = \mathrm{tr}_A(\rho(A_a^x \otimes \mathbb{1})) \quad \text{(unnormalized: } \mathrm{tr}(\rho(a|x)) = P(a|x))$$

- "Steering" of Bob's system by Alice.
- We call the set of unnormalized states $\rho(a|x)$ **steering data**.
- Steering data is a quantum analogue of the conditional probability distribution $P(a|x)$:

$$\text{classical: } P(a|x) \rightsquigarrow \text{quantum: } \rho(a|x)$$

Theorem

Steering data $\rho(a|x)$ can arise in this way if and only if it satisfies the no-signaling condition:

$$\sum_a \rho(a|x) \quad \text{is independent of } x\,.$$

Bipartite steering data I

- Similarly, one can consider the case where both Alice and Bob measure and can steer the system of a third party:

$$\rho(a|x) = \mathrm{tr}_{AB}(\rho(A_a^x \otimes \mathbb{1} \otimes \mathbb{1})), \qquad \rho(b|y) = \mathrm{tr}_{AB}(\rho(\mathbb{1} \otimes B_b^y \otimes \mathbb{1})).$$

- The system of the third party is taken to be of fixed dimension d, so that $\rho(a|x), \rho(b,y) \in M_d(\mathbb{C})$.
- This is **bipartite steering data**.
- No joint measurements needed: $\rho(a|x)$ and $\rho(b|y)$ are quantum analogues of the marginals $P(a|x)$ and $P(b|y)$!
- Again, there is an obvious variant of Tsirelson's problem.

Bipartite steering data II

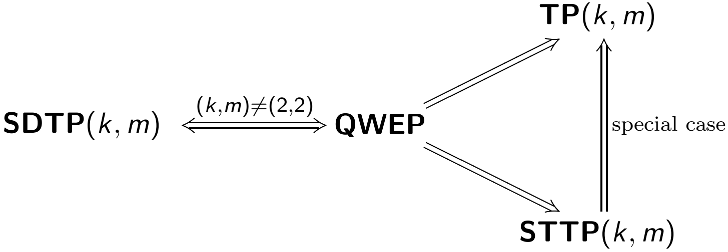
Theorem

1. If QWEP holds, then Tsirelson's problem for bipartite steering data has a positive answer in all Bell scenarios.
2. If Tsirelson's problem for bipartite steering data has a positive answer in **some** (non-CHSH) Bell scenario, then QWEP holds.

$\rightarrow$ Tsirelson's problem for bipartite steering data can be considered a physical reformulation of the QWEP conjecture.

Diagram of implications

Bipartite Bell scenario with k measurements per party and m outcomes each.



Summary

- We replace the tensor product axiom of quantum mechanics by the commutativity assumption and study whether this allows more nonlocality ("Tsirelson's problem").
- Assuming the validity of Kirchberg's QWEP conjecture, we find a positive answer to this. Likewise for Tsirelson's problem on spatiotemporal quantum correlations.
- The proof is based on relating quantum correlations to group C^*-algebras.
- Tsirelson's problem for bipartite steering data is equivalent to QWEP for every (non-CHSH) Bell scenario.
- In particular, a proof or counterexample in any scenario would at the same time decide the problem for all other scenarios.

The End

... or rather the beginning?

Backup slides

Proving nonlocality by spatiotemporal correlations I

- 2 qubits on Alice's side, 1 qubit on Bob's side. W state:

$$|W\rangle = \frac{1}{\sqrt{3}}|00\rangle \otimes |1\rangle + \frac{1}{\sqrt{3}}\left(|01\rangle + |10\rangle\right) \otimes |0\rangle \, .$$

- ± 1-valued observables:

$$A_1 = \sigma_z \otimes \mathbb{1}, \quad A_2 = \sigma_x \otimes \mathbb{1}, \quad A_3 = \mathbb{1} \otimes \sigma_z, \quad A_4 = \mathbb{1} \otimes \sigma_x \, ,$$

$$B_1 = \frac{\sigma_z - \sigma_x}{\sqrt{2}}, \qquad B_2 = \frac{\sigma_z + \sigma_x}{\sqrt{2}} \, .$$

Proving nonlocality by spatiotemporal correlations II

- If Alice measures only once: A_3 and A_4 are redundant, since they yield the same joint statistics as A_1 and A_2.
  $\Rightarrow$ Situation is equivalent to a 2-qubit system in the CHSH scenario.
  By calculation: no CHSH violation.
- When Alice is allowed to measure twice: she begins with A_1 and then chooses between A_3 and A_4. The ensuing correlations show a Hardy-type nonlocality.

It's a bad example: Alice's sequential measurements commute.
Open problems:

- Find better examples!
- Are there examples where the correlations are local with up to n sequential measurements, but nonlocal with $n+1$ sequential measurements?

The multipartite Tsirelson's problem

- Tsirelson's problem, its two extensions, and the QWEP conjecture generalize all to the multipartite case.
- It seems that the same considerations as in the bipartite case imply the same results; details will have to checked.
- Even if the usual QWEP conjecture is true, the multipartite Tsirelson's problem may still be false.