Large violation of Bell inequalities with low entanglement

M. Junge and C. Palazuelos

Department of Mathematics, University of Illinois at Urbana-Champaign, Illinois 61801-2975, USA

In this work we show the existence of violations of general bipartite Bell inequalities of order $\frac{\sqrt{n}}{\log n}$ with n inputs, n outputs and n-dimensional Hilbert spaces. Analyzing the construction of the elements involved in this violation one finds that, even though entanglement is necessary to obtain violation of Bell inequalities, the entropy of entanglement of the underlying state is essentially irrelevant in obtaining large violation. We also indicate why the maximally entangled state is a rather poor candidate in producing large violations with arbitrary coefficients. However, we also show that for Bell inequalities with positive coefficients (in particular, games) the maximally entangled state achieves the largest violation up to a logarithmic factor.

INTRODUCTION

Bell inequalities were originally proposed by Bell in 1964 as a way of testing the validity of Einstein- Podolski-Rosen's believe that local hidden variable models are a possible underlying explanation of physical reality [7]. Bell showed that the assumption of a local hidden variable model implies some inequalities on the set of probabilities, since then called *Bell inequalities*, which are violated by certain quantum probabilities produced with an *entangled state* [2]. For a long time after this, entanglement and violation of Bell inequalities were thought to be parts of the same concept. This changed in the late 1980s with a number of surprising results ([16], [13], [8]) which showed that, although entanglement is necessary for the violation of Bell inequalities, the converse is not true. On the other hand, we must point out that violation of Bell inequalities is the only way to detect entanglement experimentally without additional hypothesis on the experiment ([1]).

Nowadays, Bell inequalities is a fundamental subject in Quantum Information Theory (QIT). Apart from the theoretical interest, Bell inequalities have found applications in many areas of QIT: quantum cryptography, complexity theory, communication complexity, estimates for the dimension of the underlying Hilbert space, entangled games, etc (see [10], [11] and the references therein). However, despite the recent research on this topic Bell inequalities and their connection to quantum entanglement have remained quite mysterious. In the few last years, the application of techniques from different areas of mathematics has started to clarify the situation. This includes the consecutive works [12] and [10] which have shown the *operator space theory* as a natural framework for the study of Bell inequalities (see also [11]). Using this connection the authors proved in [12] the existence of unbounded violations of tripartite correlation Bell inequalities, answering an old question stated by Tsirelson [15]. Moreover, in [10] the authors used operator spaces techniques to get unbounded violations of general bipartite Bell inequalities.

BELL INEQUALITIES

A standard scenario to study quantum nonlocality consists in two spatially separated and non communicating parties, usually called Alice and Bob. Each of them can choose among different observables, labelled by $x = 1, \cdots, N$ in the case of Alice and $y = 1, \cdots, N$ in the case of Bob. The possible outcomes of this measurements are labelled by $a = 1, \cdots, K$ in the case of Alice and $b = 1, \cdots, K$ in the case of Bob. For fixed x, y, we will consider the probability distribution $(P(a, b|x, y))_{a,b=1}^K$. Actually, the collection $P = (P(a, b|x, y))_{x,y;a,b=1}^{N,K} \in \mathbb{R}^{N^2K^2}$ will be also called *probability distribution.*

We say that a probability distribution P is *LHV (Local Hidden Variable)* if

$$P(a, b|x, y) = \int_\Omega P_\omega(a|x)Q_\omega(b|y)d\mathbb{P}(\omega)$$

for every x, y, a, b, where $(\Omega, \Sigma, \mathbb{P})$ is a probability space, $P_\omega(a|x) \geq 0$ for all a, x, ω, $\sum_a P_\omega(a|x) = 1$ for all x, ω and the analogous conditions for $Q_\omega(b|y)$. We denote the set of LHV probability distributions by $\mathcal{L}$. We say that P is *Quantum* if there exist two Hilbert spaces H_1, H_2 such that

$$P(a, b|x, y) = tr(E_x^a \otimes F_y^b \rho)$$

for every x, y, a, b, where $\rho \in B(H_1 \otimes H_2)$ is a density operator and $(E_x^a)_{x,a} \subset B(H_1)$, $(F_y^b)_{y,b} \subset B(H_2)$ are two sets of operators representing POVM measurements on Alice and Bob systems. We denote the set of quantum probability distributions by $\mathcal{Q}$. It is well known (see [15]) that $\mathcal{L} \subsetneq \mathcal{Q} \subset \mathcal{R} = \mathbb{R}^{N^2K^2}$.

Since $\mathcal{L}$ is a polytope it can be characterized by a finite set of linear inequalities (the so called *Bell inequalities*). In general we assign a Bell inequality to every linear functional M in (the dual of) $\mathcal{R}$:

$$\text{For every } P \in \mathcal{L}, \ |\sum_{x,y;a,b=1}^{N,K} M_{x,y}^{a,b} p(a, b|x, y)| \leq C.$$

We can shorten the above notation by writing $|\langle M, P\rangle| \leq C$ and we will simply refer to the functional M as a

Bell inequality, assuming that C is defined by $C = \sup_{P\in\mathcal{L}} |\langle M, P\rangle|$.

Since $\mathcal{L} \subsetneq \mathcal{Q}$, Quantum Mechanics allows for a violation of at least some of these inequalities. We define the *violation of a Bell inequality M by a distribution Q* as

$$\frac{|\langle M, Q\rangle|}{\sup_{P\in\mathcal{L}} |\langle M, P\rangle|}.$$

In order to measure the *largest quantum violation*, we are interested in the numbers:

$$\nu(Q) = \sup_M \frac{|\langle M, Q\rangle|}{\sup_{P\in\mathcal{L}} |\langle M, P\rangle|} \tag{1}$$

and

$$\sup_{Q\in\mathcal{Q}} \nu(Q). \tag{2}$$

Beyond the theoretical interest of (1) and (2) as a measure of nonlocality, these terms turn out to be a very useful measure regarding the applications in different contexts. Indeed, in [11] (see also [10], [6]) the authors showed their immediate application to dimension witness, communication complexity or entangled games. Moreover, via a reformulation of $\nu(Q)$, this term can be shown to be very useful to measure nonlocality in the presence of noise or/and detector inefficiencies (see [10], Section 5). This is the key point in the search of a loophole free Bell test.

The following theorem allows us to translate the physical problem to the language of operator spaces.

Theorem 1 ([10], [9]). *If we are dealing with N inputs and K outputs we have*

$$\sup_{Q\in\mathcal{Q}} \nu(Q)$$

$$\simeq \|id \otimes id : \ell_1^N(\ell_\infty^K) \otimes_\epsilon \ell_1^N(\ell_\infty^K) \to \ell_1^N(\ell_\infty^K) \otimes_{min} \ell_1^N(\ell_\infty^K)\|.$$

Here $\simeq$ denotes equality up to a universal constant and ϵ and min are the smallest tensor norms in the Banach space category and operator space category respectively.

UNBOUNDED VIOLATION AND QUANTUM ENTANGLEMENT

The main result of the presented work can be stated as follows:

Theorem 2 (Theorem 1.2, [9]). *For every $n \in \mathbb{N}$ there exit a Bell inequality $M = (M_{x,y}^{a,b})_{x,y,a,b=1}^n$ and some POVMs $\{E_x^a\}_{x,a=1}^n$ such that for any diagonal pure state $|\psi\rangle = \sum_{i=1}^n \alpha_i |ii\rangle$ ([17]), we have*

$$\nu(Q_{|\psi\rangle}) \geq \frac{|\langle M, Q_{|\psi\rangle}\rangle|}{\sup_{P\in\mathcal{L}} |\langle M, P\rangle|} \succeq \frac{1}{\log n} \alpha_1 (\sum_{i=2}^n \alpha_i),$$

where $Q_{|\psi\rangle}(a, b|x, y) = \langle\psi|E_x^a \otimes E_y^b|\psi\rangle$ for every $x, y, a, b = 1, \cdots, n$.
Here, we use $\succeq$ to denote inequality up to a universal constant independent of $n \in \mathbb{N}$.

As an immediate consequence we obtain

Corollary 3. *For every $n \in \mathbb{N}$ there exists a quantum probability distribution Q with n inputs, n outputs and Hilbert spaces of dimension n such that*

$$\nu(Q) \succeq \frac{\sqrt{n}}{\log n}.$$

Corollary 3 means an improvement of all previous results about unbounded violation of Bell inequalities. Specifically, the most important improvement lies in the use of a polynomial number of inputs to obtain such a violation (compare to the most recent results [10] and [5] where an exponential number of inputs where required in order to obtain similar unbounded violations). This point is specially relevant regarding the possible applications of Theorem 2. Furthermore, we showed

Theorem 4 (Theorem 6.8, [9]).

$$\nu(Q) \preceq \min\{N, K, d\}$$

for every quantum probability distribution Q constructed with N inputs, K outputs and Hilbert spaces of dimension d.

Thus, Theorem 2 almost closes the gap with the upper bounds in all the parameters of the problem (N, K, d).

However, the main point of Theorem 2 is that it provides a lower bound for the violation of Bell inequalities that a given bipartite pure state may attain as a function of its eigenvalues. This allowed us to study the connection between two concepts at the heart of QIT: violation of Bell inequalities and quantum entanglement. In particular, if we denote by $\mathcal{E}(|\psi\rangle)$ the *entropy of entanglement* of the pure state $|\psi\rangle$, we obtained:

Corollary 5 (Corollary 1.3, [9]). *For any $\delta > 0$ we can find a quantum pure state $|\psi_\delta\rangle$ in a high enough dimension n with entanglement verifying: $\log n - \mathcal{E}(|\psi_\delta\rangle) < \delta$ (resp. $\mathcal{E}(|\psi_\delta\rangle) < \delta$) and such that*

$$\nu(Q_{|\psi_\delta\rangle}) \geq \frac{|\langle M, Q_{|\psi_\delta\rangle}\rangle|}{\sup_{P\in\mathcal{L}} |\langle M, P\rangle|} \succeq \frac{\sqrt{n}}{(\log n)^2}.$$

Corollary 5 tells us that *even though quantum entanglement is needed to obtain violation of Bell inequalities, the amount of entanglement is essentially irrelevant for large violation.* Indeed, we can find states with entropy of entanglement close to either 0 or $\log n$ and this only decreases violation by a logarithmic factor.

THE ROLE OF THE MAXIMALLY ENTANGLED STATE

It is interesting to note that the construction in Theorem 2 doesn't say anything about the extremal cases: entanglement 0 (which is trivial) and maximal entanglement. This led us to the following result:

Theorem 6 (Theorem 1.4, [9])**.** *There exists a Bell inequalities M with 2^{n^2} inputs and n outputs with the following properties:*

a) There exists a quantum probability distribution Q constructed with Hilbert spaces of dimension n such that

$$\frac{|\langle M, Q\rangle|}{\sup_{P\in\mathcal{L}}|\langle M, P\rangle|} \succeq \frac{\sqrt{n}}{\log n}.$$

b) $\sup\{|\langle M, Q_{max}\rangle|\} \leq 1$*, where this* sup *runs over all quantum probability distributions Q_{max} constructed with the maximally entangled state in* any *dimension.*

In particular, Theorem 6 shows the existence of quantum probability distributions Q which can not be written as a quantum probability distribution by using the maximally entangled state, even when the dimension of the Hilbert spaces is not restricted (note the difference with the case of quantum correlations matrices, [15]). On the other hand, Theorem 6 suggests that the maximally entangled state is a poor candidate to get large violations. A similar statement holds in the context of tripartite correlations (see [12] and the recent generalization to diagonal states in [4]). However, in [5] the authors have shown the existence of a Bell inequality for which the maximally entangled state in dimension n gives violations of order $\frac{\sqrt{n}}{\log n}$. Therefore, we can not expect to have condition b) in Theorem 6 for every Bell inequality M.

However we showed that Theorem 6 is not longer true if we restrict to Bell inequalities with positive coefficients (in particular, games), because in that case we have:

Theorem 7 (Theorem 5.7, [9])**.** *Let $M = (M_{x,y}^{a,b})_{x,y,a,b}$ be a Bell inequality with positive coefficients. Let's assume that there exists a state ρ acting on a n-dimensional Hilbert space H and verifying $|\langle M, Q_\rho\rangle| = C$, where Q_ρ denotes any quantum probability distribution constructed with the state ρ. Then, there exists $k \leq n$ such that*

$$|\langle M, Q_{|\psi_k\rangle\langle\psi_k|}\rangle| \geq \frac{C}{4\log n},$$

where $|\psi_k\rangle = \frac{1}{\sqrt{k}}\sum_{i=1}^k |ii\rangle$ is the maximally entangled state in dimension k.

It is interesting to point out that, as a consequence of the Raz parallel repetition theorem [14] and the existence of pseudo-telepathy games [3] one can deduce the existence of games which give unbounded violation of polynomial order in the dimension (although the best known estimates are very far from the order $\sqrt{n}$). This polynomial order makes the logarithmic factor appearing in Theorem 7 essentially irrelevant in our context.

The authors are partially supported by National Science Foundation grant DMS-0901457.

[1] A. Acin, N. Gisin, L. Masanes, *From Bells Theorem to Secure Quantum Key Distribution*, Phys. Rev. Lett. 97, 120405 (2006).
[2] J.S. Bell, *On the Einstein-Poldolsky-Rosen paradox*, Physics, **1**, 195 (1964).
[3] G. Brassard, A. Broadbent, A. Tapp, *Quantum Pseudo-Telepathy* Foundations of Physics, Volume 35, Issue 11, 1877 - 1907 (2005).
[4] J. Briet, H. Buhrman, T. Lee, T. Vidick, Multiplayer XOR games and quantum communication complexity with clique-wise entanglement, arXiv:0911.4007
[5] H. Buhrman, G. Scarpa, R. de Wolf, *Better Non-Local Games from Hidden Matching*, arXiv:1007.2359.
[6] J. Degorre, M. Kaplan, S. Laplante, J. Roland, *The communication complexity of non-signaling distributions*, Proc. 34th Int. Symp. of the MFCS, 270-281 (2009).
[7] A. Einstein, B. Podolsky, N. Rosen, *Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?*, Phys. Rev., **47**, 777 (1935).
[8] N. Gisin, *Hidden quantum nonlocality revealed by local filters*, Phys. Lett. A 210, 151 (1996).
[9] M. Junge, C. Palazuelos, *Large violation of Bell inequalities with low entanglement*, arXiv:1007.3043.
[10] M. Junge, C. Palazuelos, D. Pérez-García, I. Villanueva and M.M. Wolf, *Unbounded violations of bipartite Bell Inequalities via Operator Space theory*, To appear in Commun. Math. Phys.
[11] M. Junge, C. Palazuelos, D. Pérez-García, I. Villanueva and M.M. Wolf, *Operator Space theory: a natural framework for Bell inequalities*, Phys. Rev. Lett. 104, 170405 (2010).
[12] Pérez-García, D., Wolf, M. M., Palazuelos, C., Villanueva, I. and Junge M., *Unbounded violation of tripartite Bell inequalities*, Commun. Math. Phys. 279 (2), 455-486 (2008).
[13] S. Popescu, *Bell's Inequalities and Density Matrices: Revealing "Hidde" Nonlocality*, Phys. Rev. Lett. 74, 2619 (1995).
[14] R. Raz, *A Parallel Repetition Theorem*, SIAM Journal on Computing 27, 763-803 (1998).
[15] B.S. Tsirelson, Hadronic Journal Supplement 8:4, 329-345 (1993).
[16] R.F. Werner, *Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model*, Phys. Rev. A 40, 4277 (1989).
[17] Note that we can always assume the pure state $|\psi\rangle$ to be diagonal when we are studying probability distributions. It suffices to consider the unitaries in the Hilbert-Schmidt decomposition as a part of the involved POVMs.