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Anyons, Twists & Topological Codes

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outline

- motivation/introduction
- twists
  - anyon models, symmetries & twists
  - toric code
- topological subsystem color codes
  - twist geometry
  - Clifford operations by code deformation
- conclusions

motivation

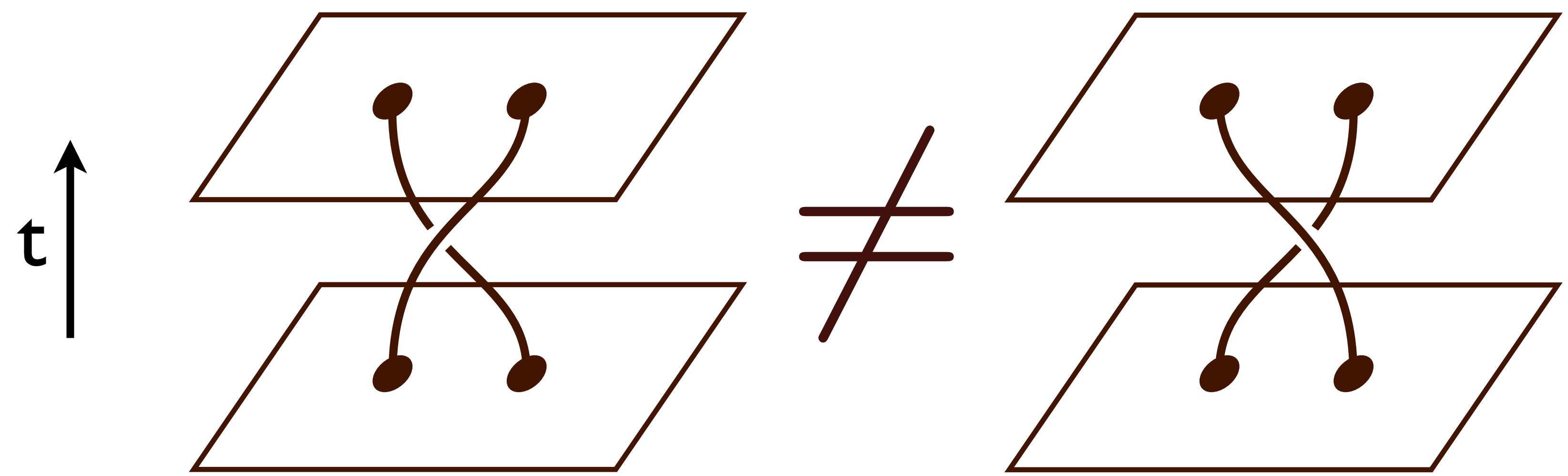
- traditional fault-tolerance is not enough to reallistically fight decoherence
- topological alternatives (Kitaev):
  - **topological quantum computing** (TQC)
    - self-protected by energy gap
    - immune to small local distortion
  - **topological codes** (TC)
    - geometrically local, active error correction
    - error threshold for large size
- both are **anyon**-based: exotic statistics in 2D

motivation

- two problems addressed here:
    - TQC: the anyons that are easier to get have no computational power
    - TC: there exist extremely local TCs (**2-local** measurements in 2D), but no way to compute
- a solution / new tool:
    - **twists** → use **anyon symmetries** to increase computational power

introduction

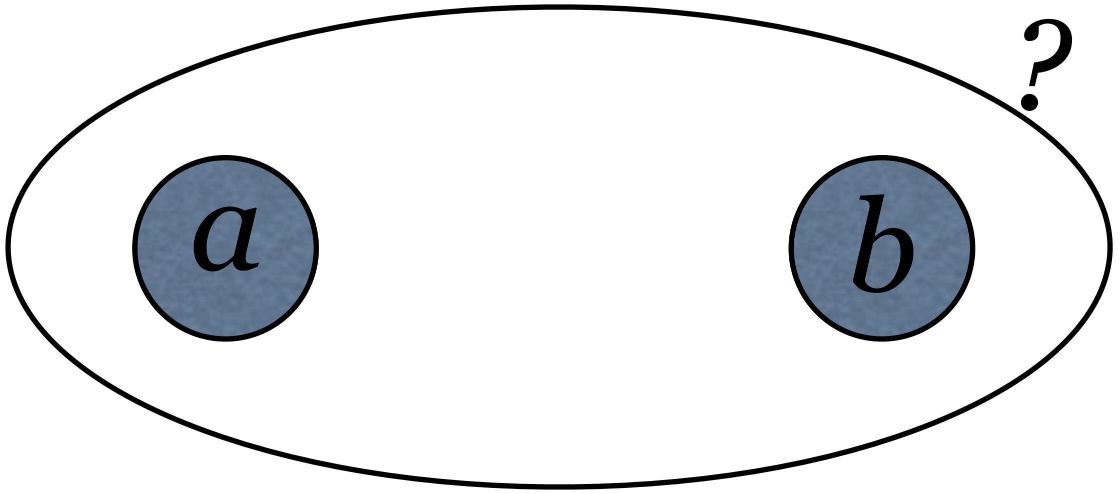
- why 2D?



- statistics beyond bosons and fermions
- topological interaction
- appear in systems with topological order (TO) (Wen '89):
    - gapped, ground state degeneracy depends on topology

introduction

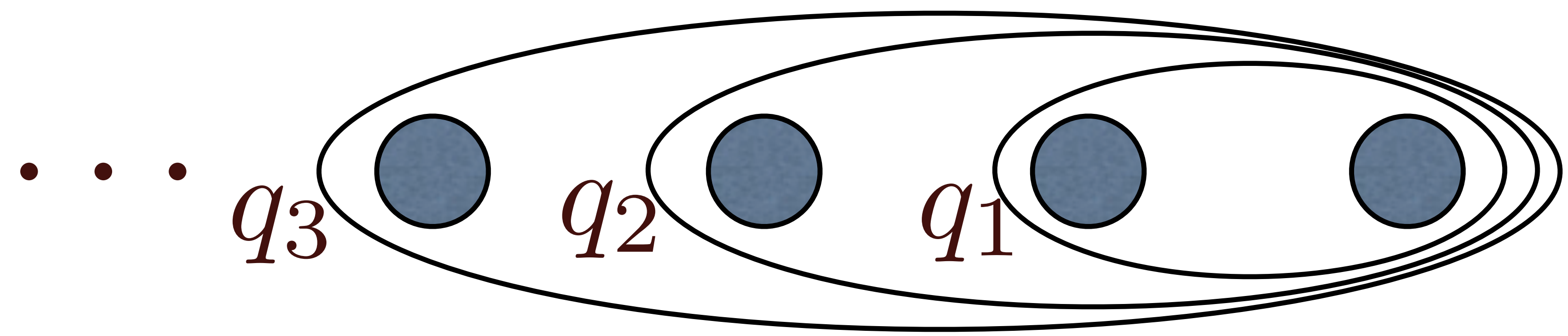
- abelian charge: given charge of constituents, total charge is known
- topological charge can be **non-abelian**



introduction

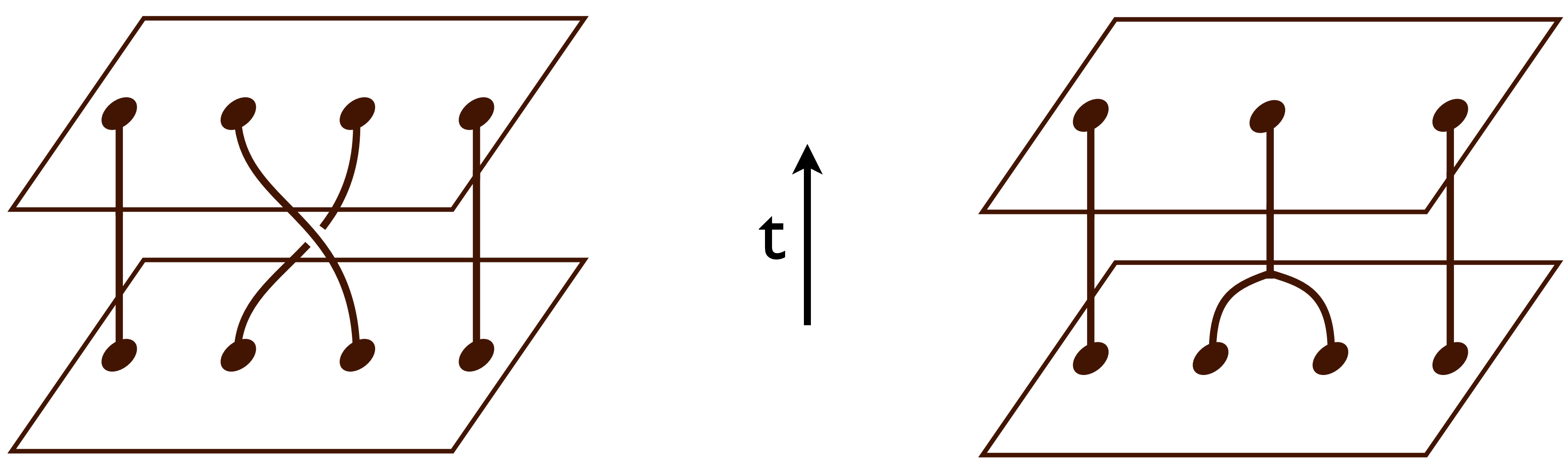
- TQC (Kitaev ’03, Freedman et al ’03)

encode in fusion channels



compute = braid

measure = fuse

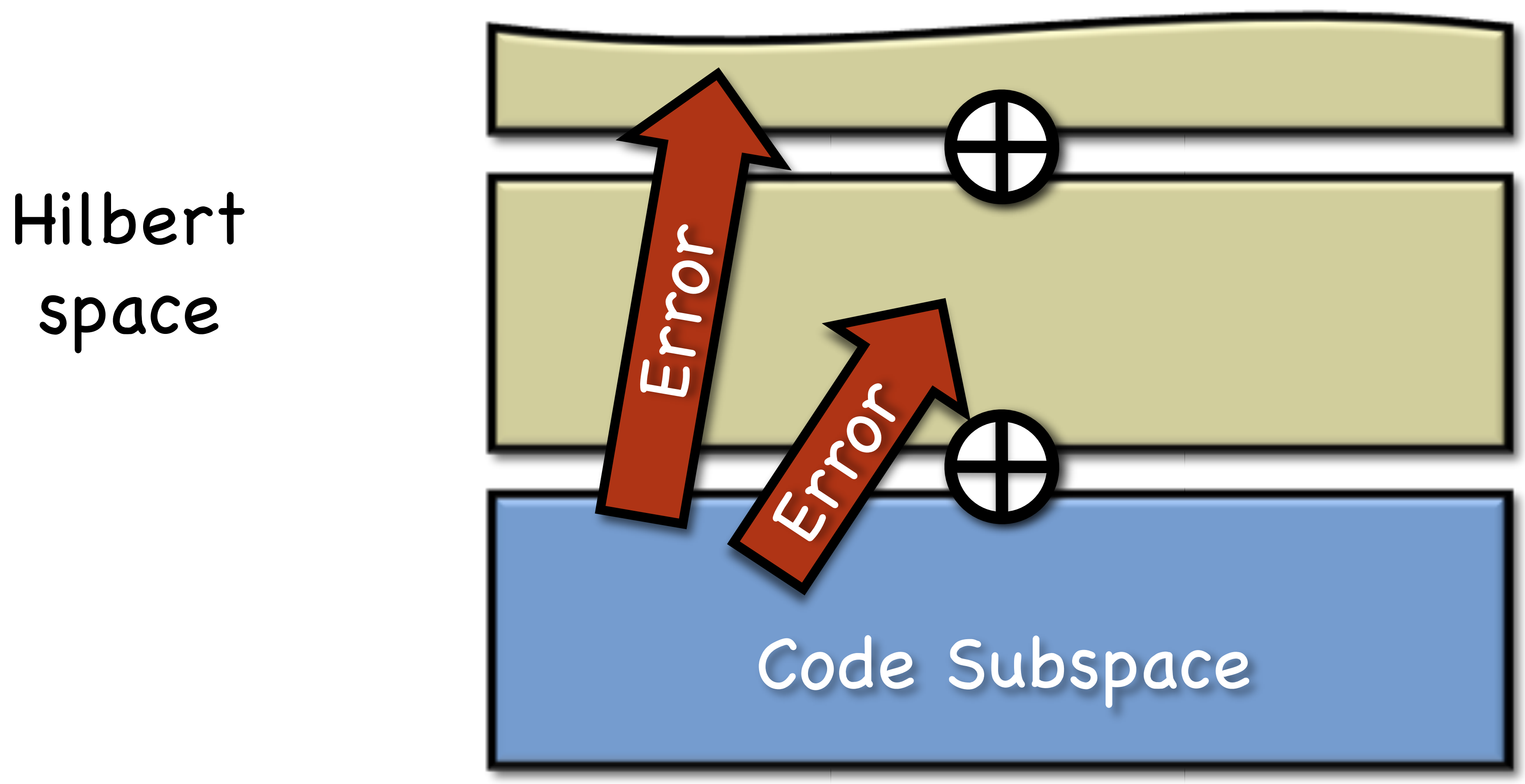


introduction

- abelian anyons have no computational power
- twists offer a way to recover non-abelian behavior!

introduction

- quantum error correcting codes protect quantum information using redundancy
- typically this involves encoding in a subspace



introduction

- the code subspace can be defined in terms of commuting observables: **check operators** (CO)

$$C_i|\psi\rangle = c_i|\psi\rangle$$

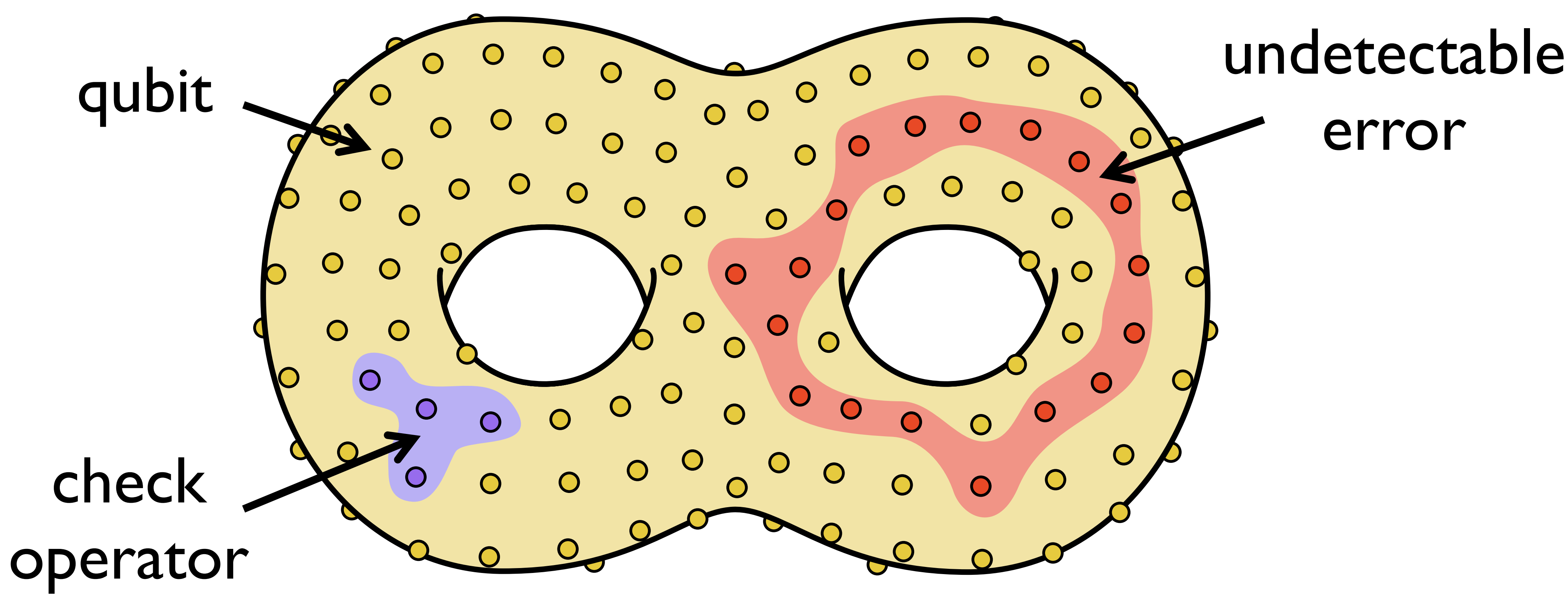
- errors typically change CO values → allows to keep track of errors

CO measurement → error syndrome →

→ compute most probable error

introduction

- topological codes (Kitaev '97)
  - geometrically **local** check operators = easy to measure
  - global undetectable errors = hard to happen

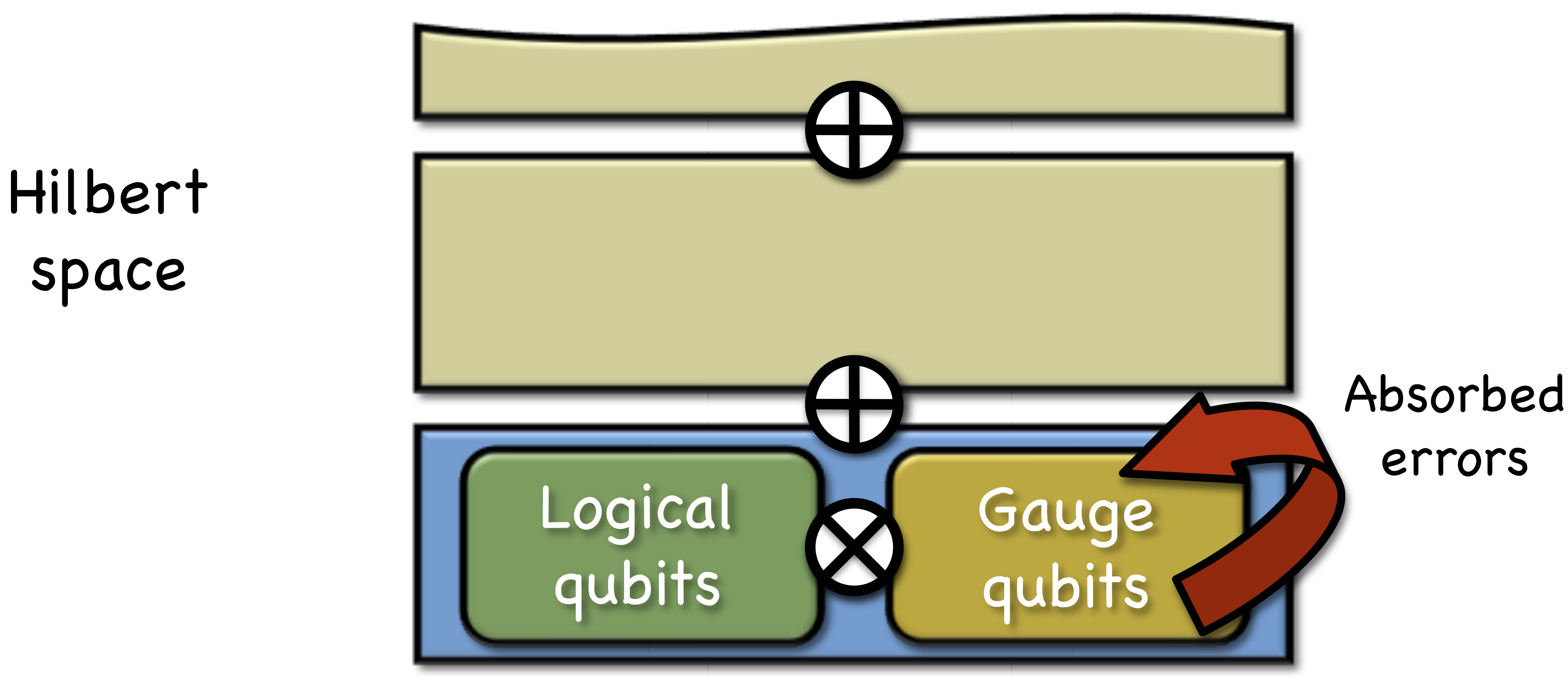


introduction

- # encoded qubits depends on topology (homology)
- flexible: many lattices allowed, transversal gates possible
- **boundaries**: planar geometries
- topological quantum memory (Dennis et al '02):
  - measure COs repeatedly
  - under a noise **threshold**, storage time exp in size
  - ideal error correction amounts to compute free energy
- **code deformation**:
  - change topology over time: initialize, compute, measure

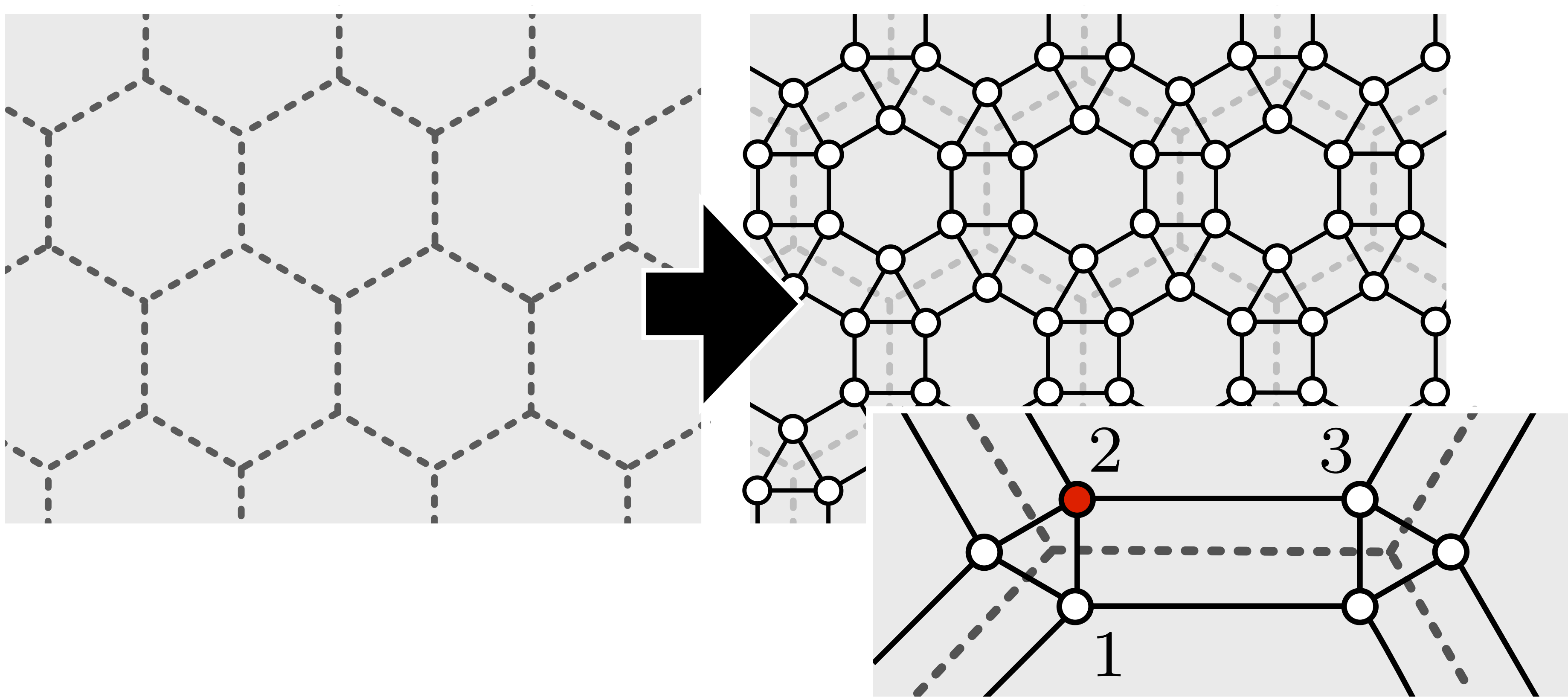
introduction

- **subsystem** codes (Kribs et al '05) can also improve locality
- only a subsystem of the code subspace is used
- check operators need not be measured directly → measurements potentially more local (Poulin '05)



introduction

- **topological subsystem color codes** (TSCC)
  (Bombin '09)
  - "doubly local": topology + subsystem
  - error syndrome recovery needs **2-local** measurements!



$Z_1 Z_2$

$Y_2 X_3$

introduction

- TC and TO are closely related for subspace codes

<u>Topological codes</u>	VS	<u>Topological order</u>
Check operators	$\leftrightarrow$	Hamiltonian terms
Code subspace	$\leftrightarrow$	Ground subspace
Error syndrome	$\leftrightarrow$	Excitation configuration

- TSCCs also have an anyonic picture for error syndromes

introduction

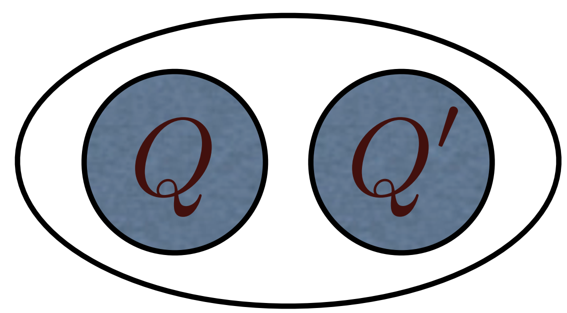
- TSCCs do not allow boundaries
  - no natural planar codes
  - code deformation becomes unpractical
- with twists
  - we can build planar TSCCs
  - whole Clifford group by code deformation!

twists

- ingredients of an anyon model:

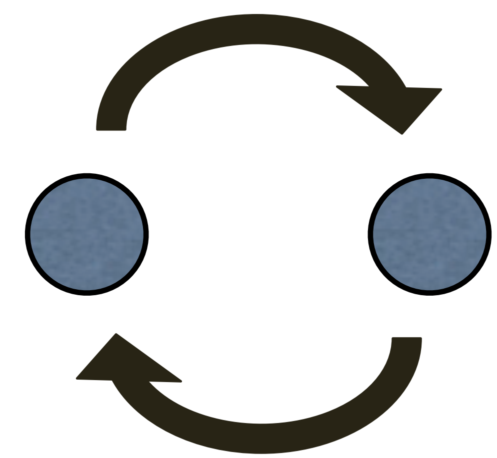
$Q \in \{a, b, \ldots\}$

topological charges



$Q \times Q' = q_1 + q_2 + \ldots$

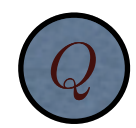
fusion rules

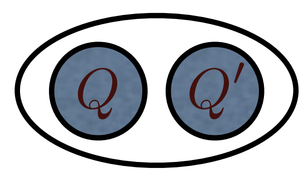


$|\psi\rangle \longrightarrow ?$

braiding rules

twists

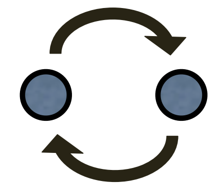
- ex.: **Ising anyons**
  - topological charges 

$\{\mathbf{1}, \sigma, \psi\}$
  - fusion rules 

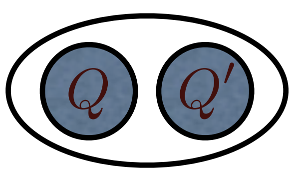
$$\sigma \times \sigma = \mathbf{1} + \psi, \qquad \sigma \times \psi = \sigma, \qquad \psi \times \psi = \mathbf{1}.$$

- the total charge of two distant σ-s is $\mathbf{1}$ or ψ:
  - if far appart, global qubit
- fusion space: $2n$ σ-s → n qubits

twists

- braiding rules: 
  - we can describe braiding up to a phase with a **Majorana operator** per σ
    - Majorana operators are self-adjoint c_i with

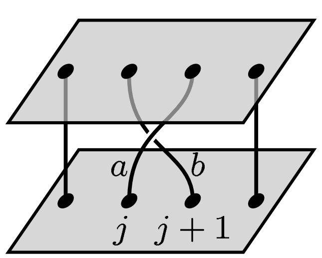
$$c_j c_k + c_k c_j = 2\delta_{jk}$$

    - total charge of j-th and j+1-th σ-s: 

$$-ic_j c_{j+1}$$

    - braiding:

$$c_j \longrightarrow c_{j+1}$$
$$c_{j+1} \longrightarrow -c_j$$



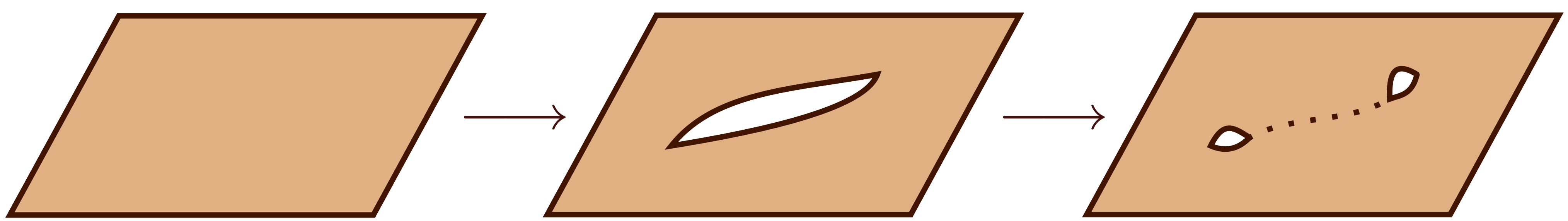
  - not universal, but we can use distillation (Bravyi '06)

twists

- **anyon symmetry**: charge permutation producing an equivalent anyon model

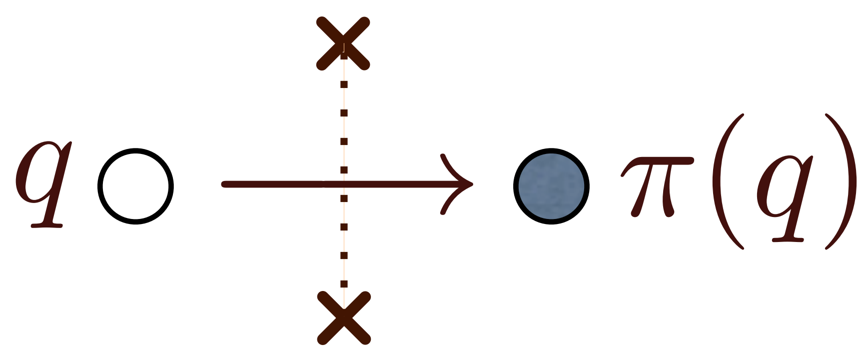
$$q \longrightarrow \pi(q)$$

- imagine ‘cutting’ the anyons’ 2D world and gluing it again up to a symmetry

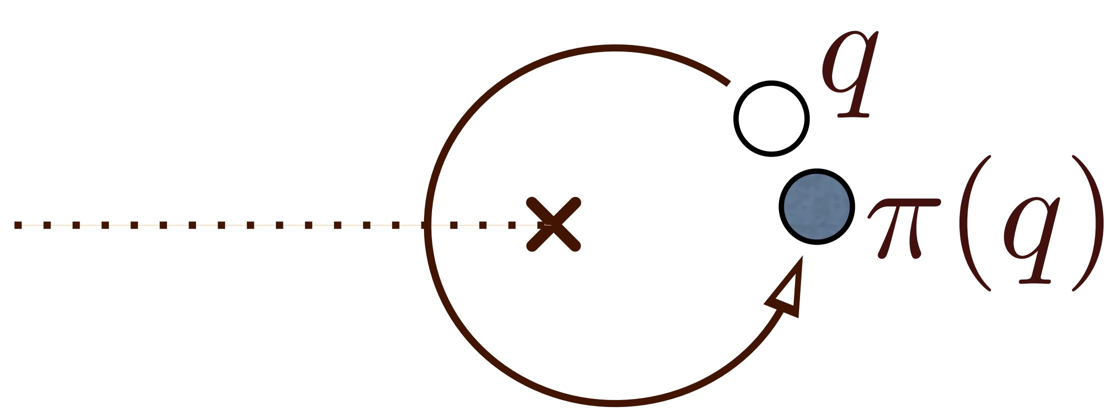


twists

- across the cut, charges change:



- topologically, the cut location is unphysical.
- endpoints are meaningful: under monodromy they permute charges → **twists**

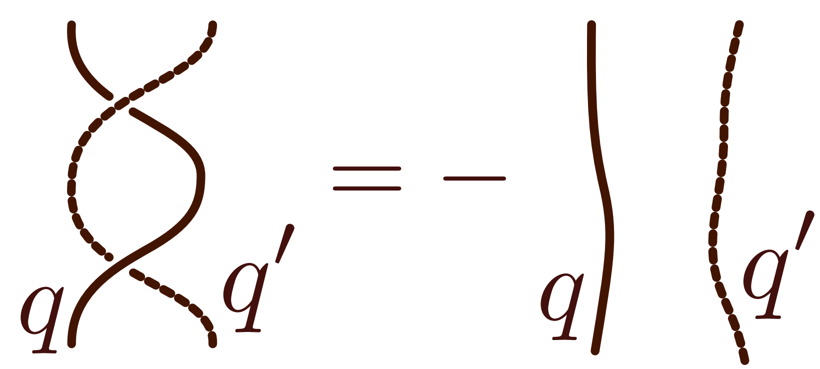


twists

- ex.: quantum double of Z_2 (**toric code**)
- charges: $\{\mathbf{1}, e, m, \epsilon\}$
- fusion: $e \times m = \epsilon \qquad e \times \epsilon = m \qquad m \times \epsilon = e$

$$e \times e = m \times m = \epsilon \times \epsilon = \mathbf{1}$$

- braiding: $e, m \longrightarrow \text{bosons} \qquad \epsilon \longrightarrow \text{fermion}$

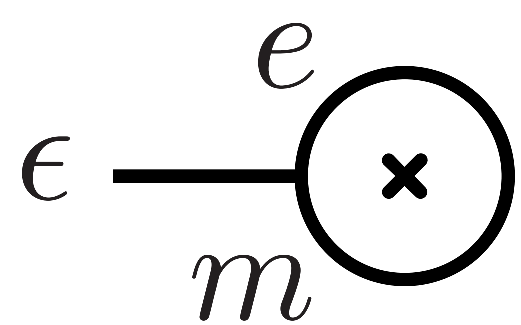


$\mathbf{1} \neq q \neq q' \neq \mathbf{1}$

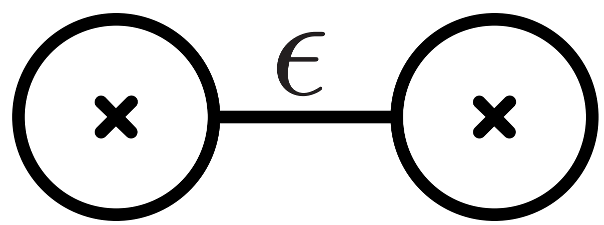
- nontrivial **symmetry**: $e \leftrightarrow m$

twists

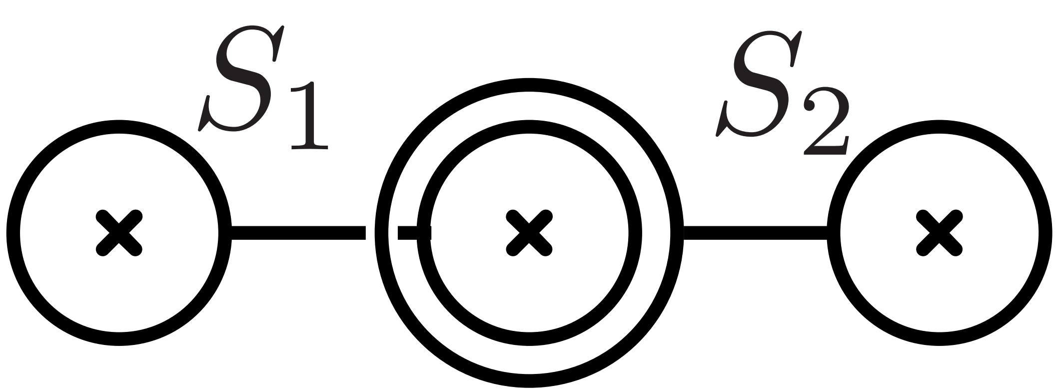
- twists are sinks/sources for fermions:



- vacuum to vacuum processes...



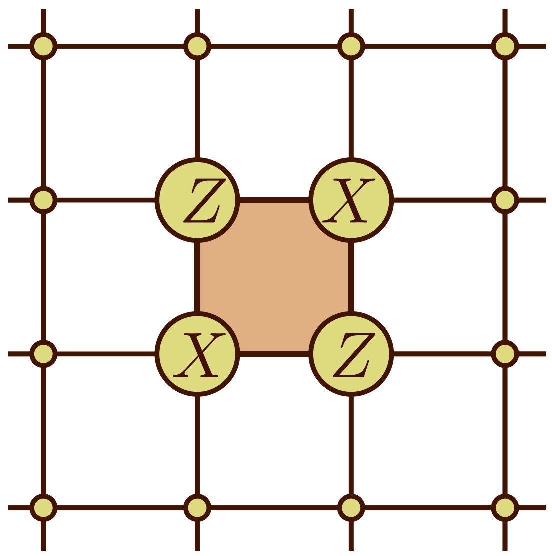
- ...lead to **topological degeneracy**:



$$S_1 S_2 = -S_2 S_1$$

twists

- toric code (Kitaev '97, Wen '03):
  - qubits form a square lattice
  - 4-local check operators at plaquettes



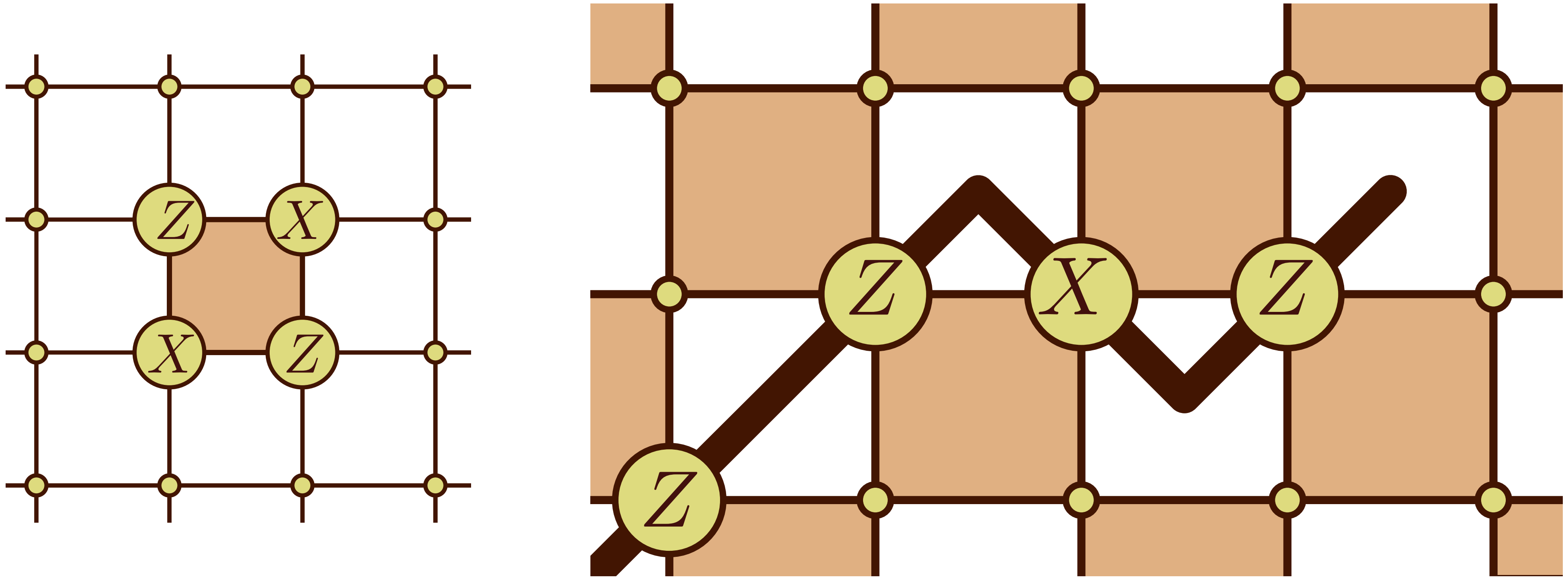
$$C_k := X_k Z_{k+\mathbf{i}} X_{k+\mathbf{i}+\mathbf{j}} Z_{k+\mathbf{j}}$$

$$C_k|\psi\rangle = |\psi\rangle$$

- Hamiltonian version: $$H := -\sum_k C_k$$
  - excitations live at plaquettes

twists

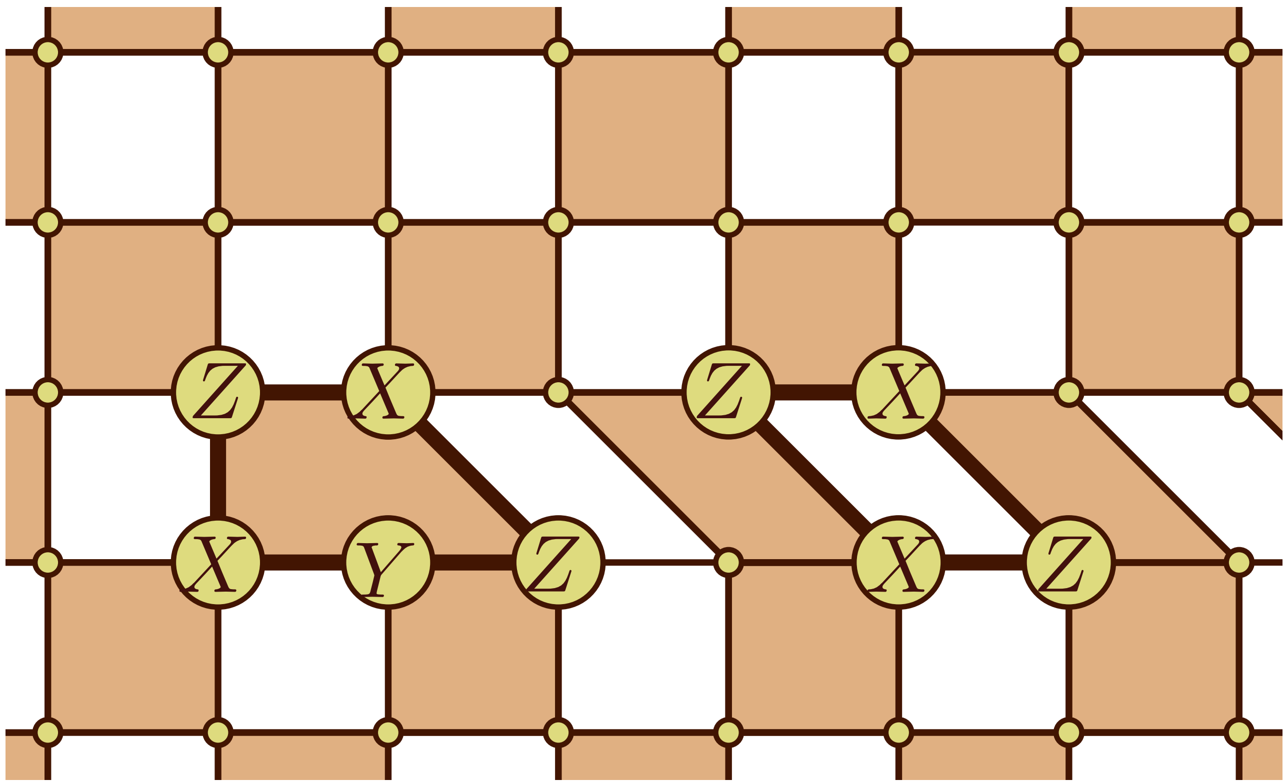
- **string operators** create/destroy excitations at their endpoints



- two types of strings/excitations: e (light) and m (dark)

twists

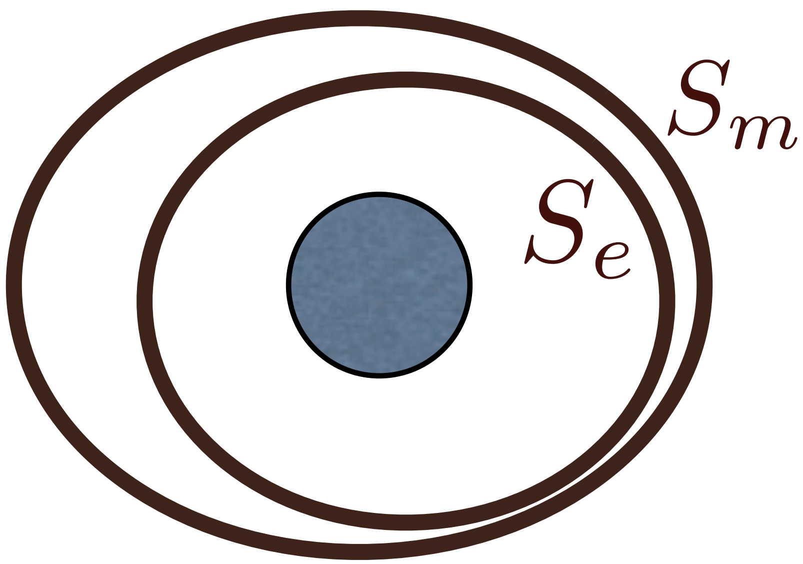
- twists amount to **dislocations**



- twists can be locally created in PAIRS only

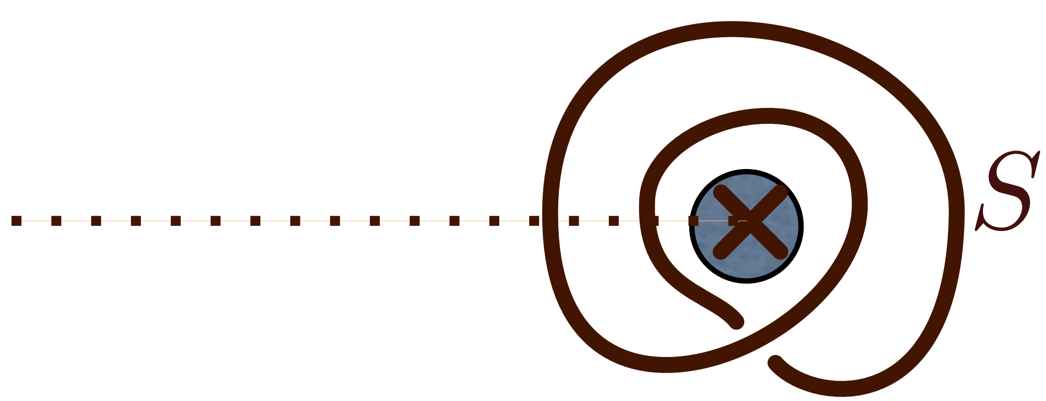
twists

- no twists (or even number) → 4 possible charges



	$\mathbf{1}$	e	m	ϵ
S_e	+1	+1	-1	-1
S_m	+1	-1	+1	-1

- a twist (or an odd number) → 2 possible charges



	σ_+	σ_-
S	+i	-i

twists

- non-abelian fusion rules!

$$\sigma_\pm \times \sigma_\pm = 1 + \epsilon \qquad \sigma_\pm \times \sigma_\mp = e + m$$

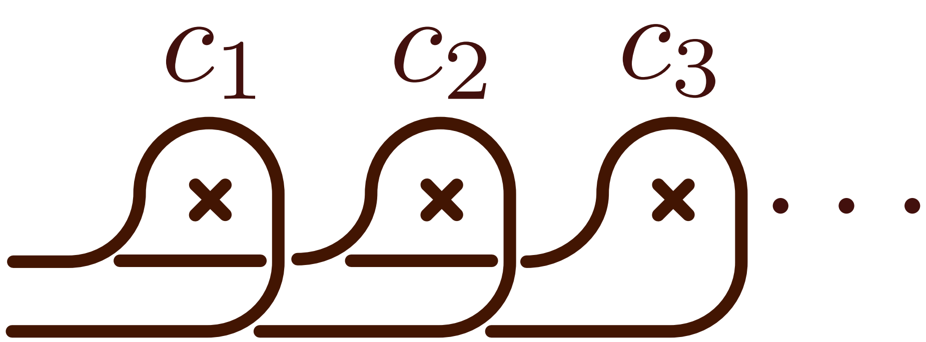
$$\sigma_\pm \times \epsilon = \sigma_\pm \qquad \sigma_\pm \times e = \sigma_\pm \times m = \sigma_\mp$$

- we recover **Ising** rules:

$$\sigma_+ \times \sigma_+ = \mathbf{1} + \epsilon \qquad \sigma_+ \times \epsilon = \sigma \qquad \epsilon \times \epsilon = \mathbf{1}$$

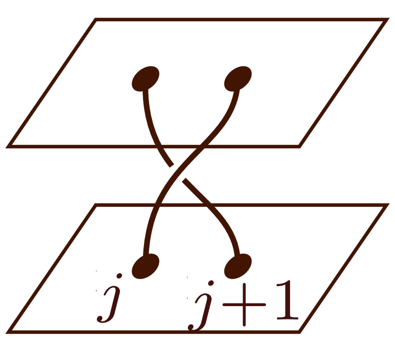
twists

- all closed string ops can be expressed in terms of a set of open string ops → Majorana operators



$$c_j c_k + c_k c_j = 2\delta_{jk}$$

- braiding is also Ising-like!



$$c_j \longrightarrow c_{j+1}$$
$$c_{j+1} \longrightarrow -c_j$$

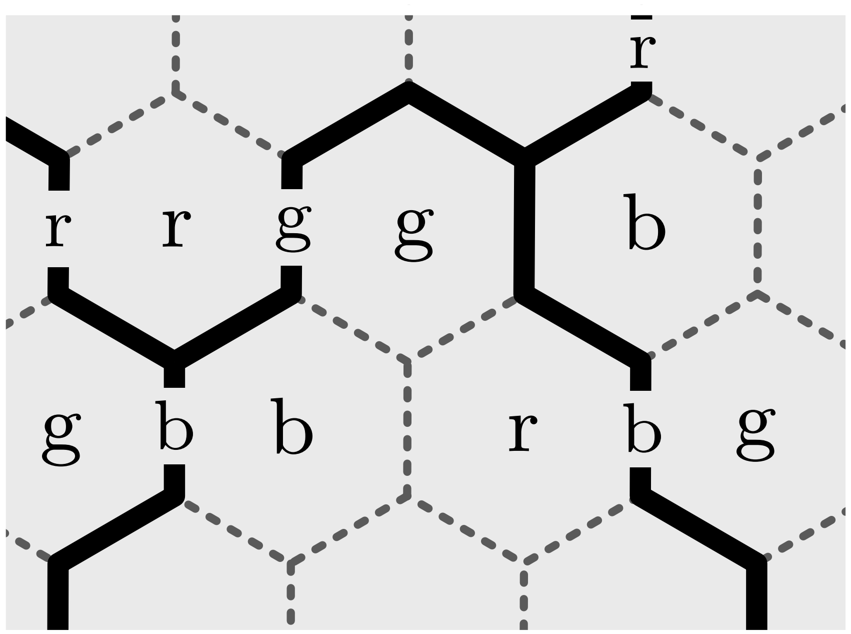
twists

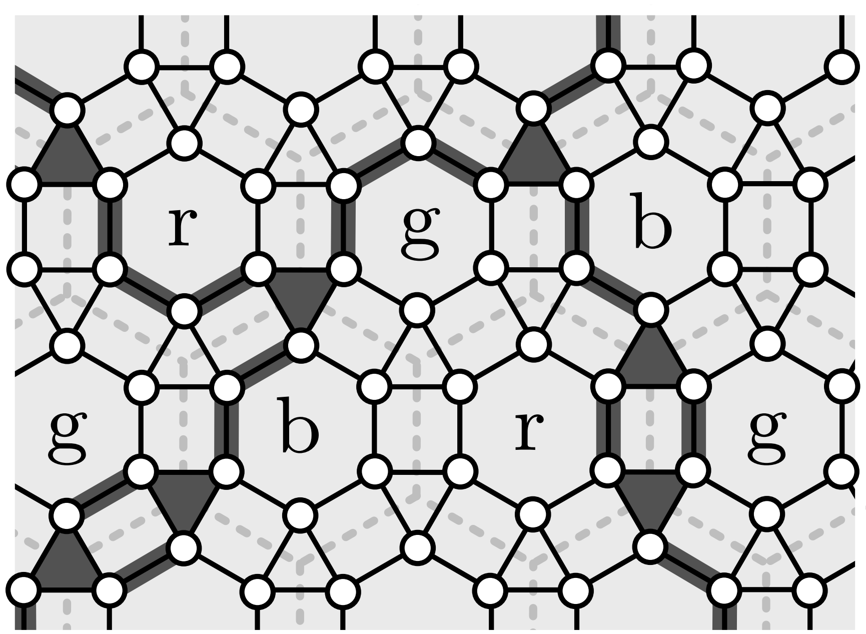
toric code $\xrightarrow{\text{twists}}$ Ising anyons

abelian $\longrightarrow$ non-abelian

TSCC

- the original TSCCs come from 3-valent lattices with **3-colorable** faces (red, green, blue)
- string operators have a color
- commutation relations of string ops relates them to an anyon model with three nontrivial charges





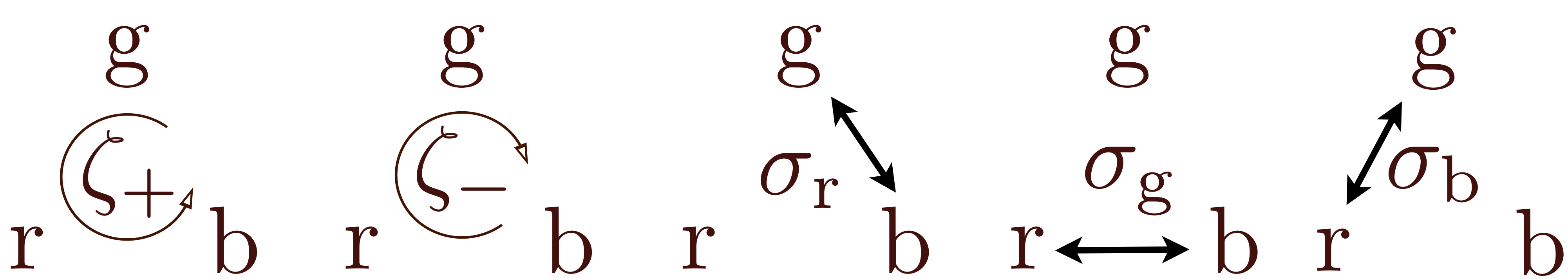
TSCC

- fusion rules as in toric code

$$\mathrm{r} \times \mathrm{g} = \mathrm{b} \qquad \mathrm{g} \times \mathrm{b} = \mathrm{r} \qquad \mathrm{b} \times \mathrm{r} = \mathrm{g}$$

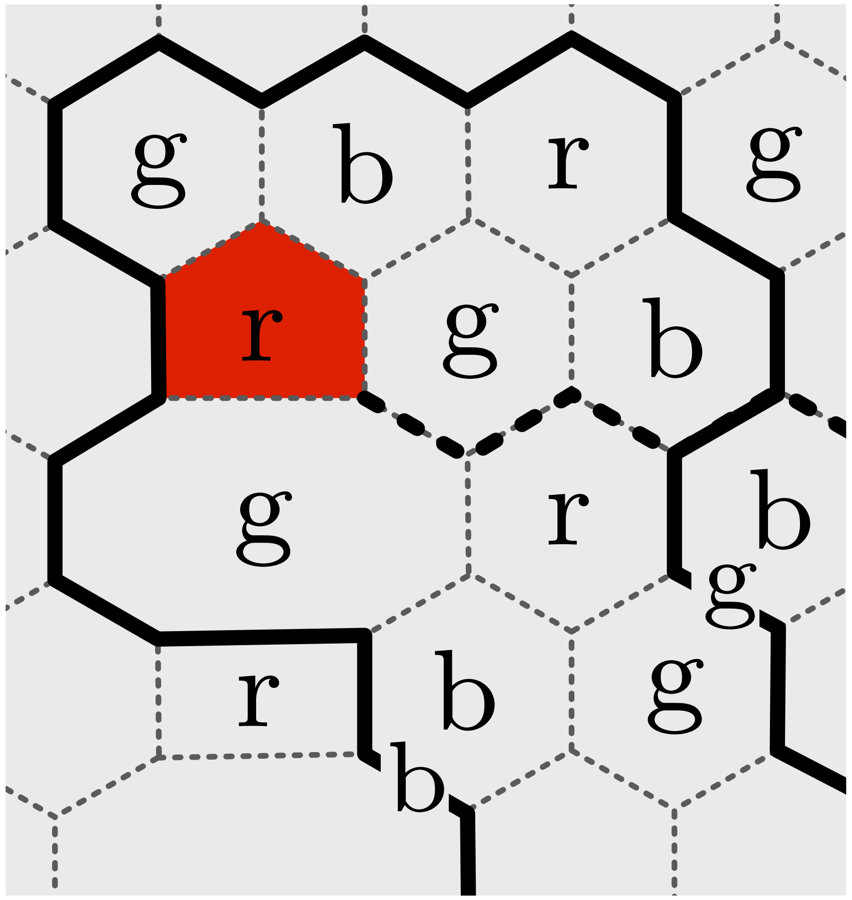
$$\mathrm{r} \times \mathrm{r} = \mathrm{g} \times \mathrm{g} = \mathrm{b} \times \mathrm{b} = \mathbf{1}$$

- braiding of different charges as in toric code
- the difference: three fermionic charges
- any **permutation** of the colors is a symmetry!
- twists are labeled by the elements of S_3



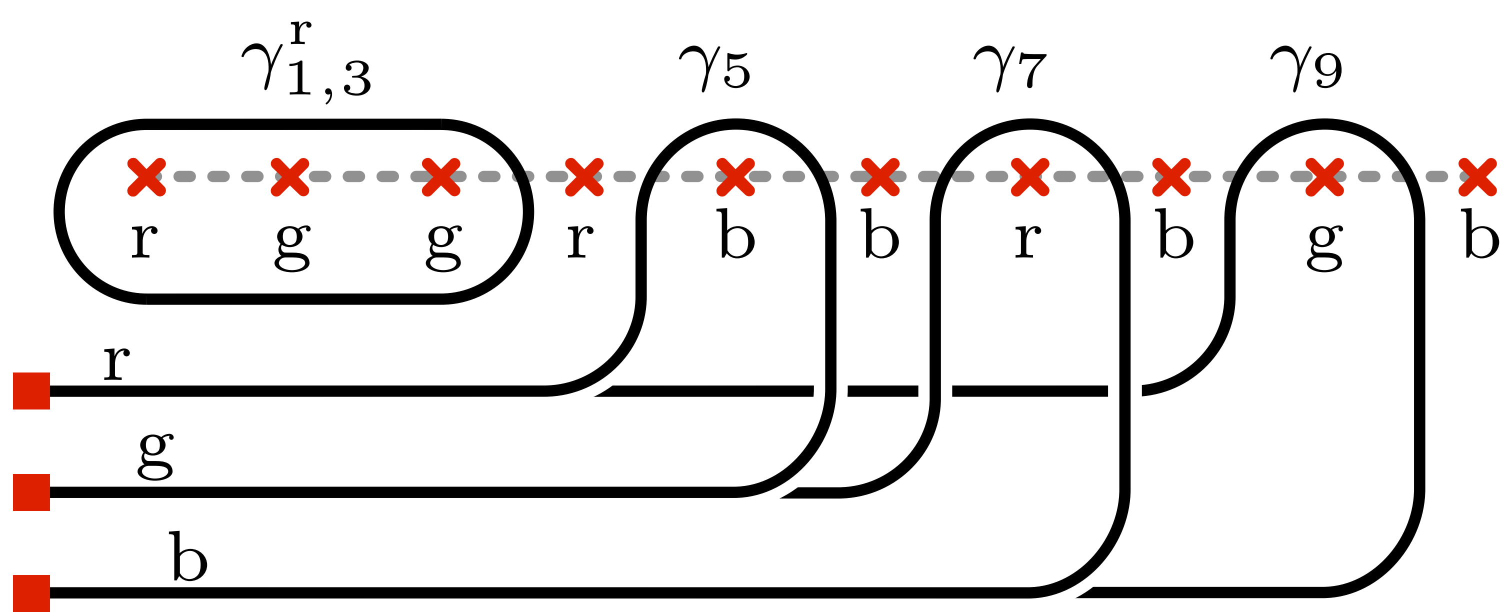
TSCC

- faces with an odd number of links brake 3-colorability
- these are twists: two colors are exchanged
- a red twist exchanges green and blue, and so on



TSCC

- to the i-th twist we attach a string γ_i...



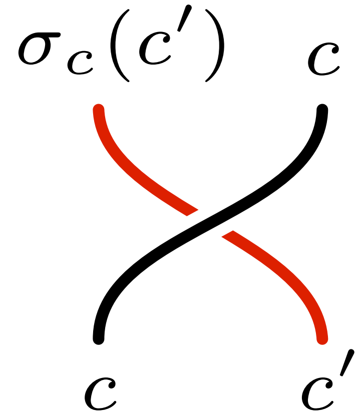
- ...and get self-adjoint string ops k_i,
  "colored" Majorana ops

$$k_i^2 = 1 \text{ and, for } i < j,$$

$$k_i k_j = \begin{cases} k_j k_i & \text{if } c_i = \zeta_+(c_j), \\ -k_i k_j & \text{otherwise.} \end{cases}$$

TSCC

- braiding changes the color of twists



- transforming as follows the colored Majorana ops (c_i is the color of the i-th twist)

$$k_j \to k_{j+1}, \quad k_{j+1} \to \begin{cases} -k_j & \text{if } c_j = c_{j+1}, \\ i k_j k_{j+1} & \text{if } c_j = \zeta_-(c_{j+1}), \\ -k_j k_{j+1} & \text{otherwise.} \end{cases}$$

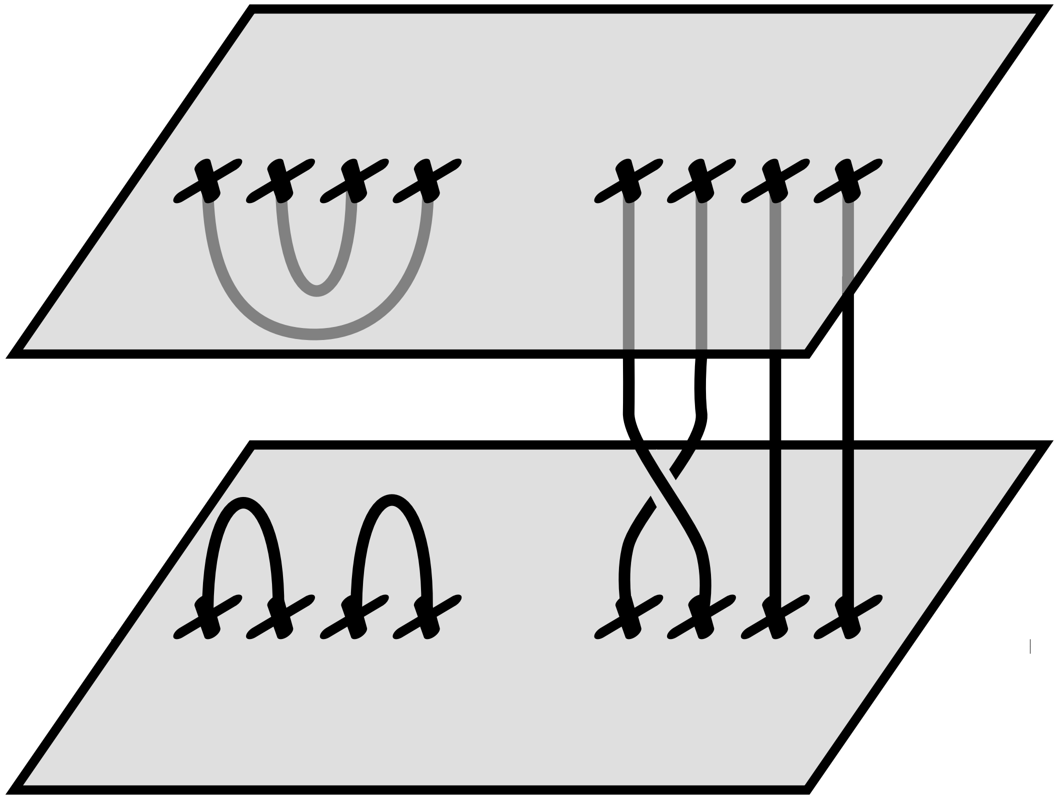
TSCC

- for twists of the same color, we are back to Ising anyons
- encoding: 1 qubit = 4 twists of the same color
- we get all single qubit Clifford gates (Bravyi '06)

$$\langle k_j k_{j+1} k_{j+2} k_{j+3} \rangle = -1$$

$$\hat{X} \equiv -i k_j k_{j+1}$$

$$\hat{Z} \equiv -i k_{j+1} k_{j+2}$$

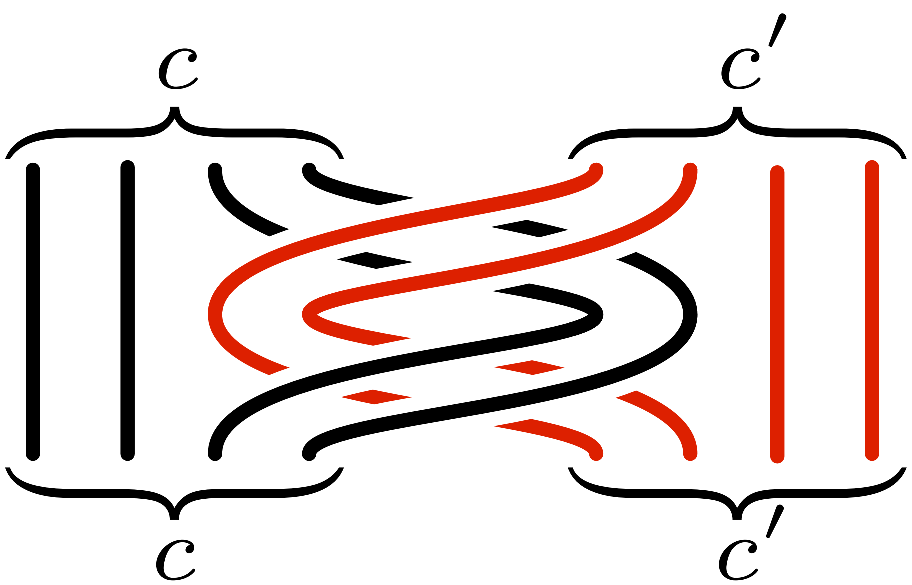


TSCC

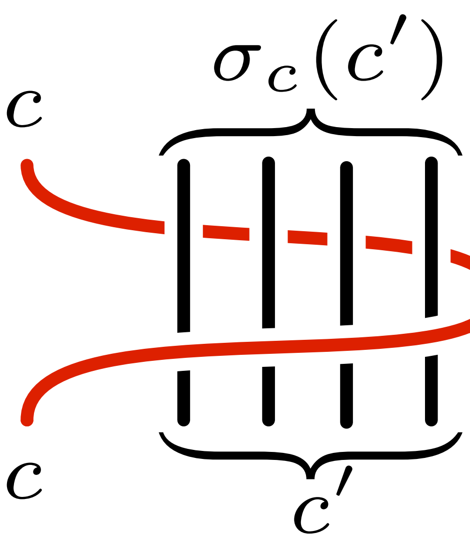
- to get the whole Clifford group, we only need to implement an **entangling gate**

- but for two groups of twists of different color:

$$\hat{X}_1 \to \hat{X}_1 \qquad \hat{Z}_1 \to \hat{X}_2\hat{Z}_1$$
$$\hat{X}_2 \to \hat{X}_2 \qquad \hat{Z}_2 \to \hat{X}_1\hat{Z}_2$$



- and we can always flip the color of a group:



TSCC

TSCC + twists +
+ code deformation =
Clifford gates

conclusions & questions

- anyon symmetries allow to introduce twists
- twists make anyon models and topological codes computationally more powerful (but how much?)
- toric codes:
  - twists mimic Ising anyons
- topological subsystem color codes:
  - Clifford operations by code deformation
- other/general anyon models?

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