Anyons, Twists and Topological Codes

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Anyons exhibit symmetries that can be exploited to increase their topological quantum computing capabilities. This is done through the introduction of twists, topological defects that realize such symmetries. Because topological codes are closely related to anyon models, twists offer also a mechanism to compute with them. In a class of topological codes where error tracking only involves measuring 2-local operators in a planar setting, twists allow the implementation of any Clifford gate by code deformations.

In order to build a quantum computer it is essential that we can address the problem of decoherence. From a theoretical perspective there exist good reasons to believe that low enough levels of noise can be managed [1, 2], but in practice fault-tolerance is difficult to achieve experimentally, at least within the 'traditional' approaches. Therefore, alternatives are needed. The present work deals with two such alternatives, *topological quantum computing* [3–5] and *topological codes* [3]. They are different in spirit, since in topological quantum computers a certain degree of protection is achieved by passive means, due to an energy gap, whereas for topological codes error correction is active as for any other class of quantum error correcting codes. However, these approaches are also closely related as both are based on *anyons*, particles with exotic statistics that might appear in systems with two spatial dimensions [6]. The present work [7, 8] introduces a tool, *twists*, that exploits anyon symmetries to increase the computational power in either of the two approaches, in some cases with dramatic effects.

What makes anyons so special is that they carry topological charge, which behaves very differently from other more conventional charges. First, topological charges can be non-abelian, in the sense that the total charge in a composite system is not fixed by the charges of its constituents. This is interesting because it gives rise to non-local degrees of freedom where quantum information can be stored. Namely, if we have several anyons far appart from each other, only their individual charge can be locally retrieved, whereas the total charge of a group of anyons might have several possible values, giving rise to an energy degeneracy that is robust under small local perturbations. Second, braiding the anyons induces a unitary transformation on these degrees of freedom, which only depends on the topology of the braid. This is how quantum gates are performed in topological quantum computing, making them naturally robust again small deformations. As for measurements, it is enough to put the anyons together, fusing them and observing the resulting outcome.

In condensed matter, anyons emerge as excitations in systems that exhibit topological order [9]. A possible way to obtain these exotic phases is by engineering suitable Hamiltonians on lattice spin systems [3, 10–13]. Indeed, implementations on optical lattices have been proposed [14]. Unfortunately, the anyon models that appear in simple models are not computationally powerful. For example, many systems are abelian, so that we cannot even use fusion channels to encode quantum information. The strategy discussed here allows to recover computationally interesting anyon-like behavior from systems with very simple anyonic statistics. In particular, twists can be created, braided and fused like anyons, yielding non-abelian behavior starting from abelian systems.

As for topological codes, their main feature with respect to general codes is their locality. Recall that an error correcting code [15, 16] protects quantum information from decoherence by means of redundancy. This is achieved by selecting a set of commuting observables, called check operators, that are initialized with a specific set of values, defining a code subspace. The idea is that most errors affect the expected value of check operators, so that a repeated measurement of check operators potentially allows to keep track of errors. In topological codes check operators are geometrically local. This is a great advantage, but other kind of codes also require only the measurement of local operators. What makes topological codes special is that for local error models they exhibit an error threshold: when noise is below the threshold, error correction is perfect in the limit of large codes. This gives rise to the concept of *topological quantum memory* [17]: by measuring check operators repeatedly, in parallel, encoded qubits can be stored almost perfectly for times exponentially large in the system size.

Topological codes have other interesting features. Error correction has a classical statistical interpretation [17–20], something that has led to very fast algorithms to compute the most probable error [20]. Qubit losses can also be naturally taken into account [21]. Because information is encoded in topological degrees of freedom, computation can be carried out by changing the code geometry over time, a technique known as *code deformation* [17, 22, 23]. Finally, error syndromes can be interpreted in terms of anyons. This is particularly important when exploring the structure of topological codes or finding extensions to their use, as the present work illustrates.

In *subsystem codes* [24] only a subsystem of the code subspace is used, so that errors that do not affect this subsystem are irrelevant. An advantage is that sometimes this makes it possible to split the measurement of check operators into more local measurements [25]. This approach was successfully applied to topological codes in [19], where a class of

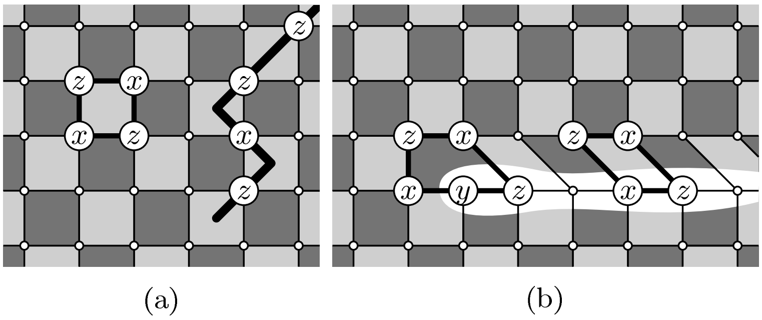

FIG. 1: A square lattice with spins living at vertices. (a) Plaquette operators are products of four Pauli operators. (b) A dislocation in the geometry of the Hamiltonian produced by shifting plaquettes. In the pentagon one can introduce the indicated plaquette operator, which commutes with the rest.

two-dimensional topological subsystem codes was introduced: *topological subsystem color codes* (TSCC). These codes only require 2-local measurements to recover the value of check operators, as simple as it may get. As a comparison, toric codes [3] require measuring 4-local operators, and for other codes things just get worse. TSCCs, however, also suffer an important disadvantage. They do not naturally admit a planar realization, making code deformations unpractical [19]. Twists overcome all this difficulties, making planar codes possible and allowing the implementation of all Clifford gates by code deformation. This is enough to perform many relevant quantum information protocols. Most importantly, it gives rise to universal quantum computation through the distillation of so called magic states [26].

Anyon symmetries and twists. Anyon models have three main ingredients: (i) a set of labels that identify the superselection sectors or topological charges, (ii) fusion/splitting rules that dictate the charges of composite systems, and (iii) braiding rules that dictate the effect of particle exchanges. A symmetry then is a permutation of the labels that leaves braiding and fusion rules unchanged. Given a symmetry s, we can imagine cutting the system along an open curve, and then gluing it again "up to s". Ideally the location of the cut itself is unphysical, only its endpoints have a measurable effect. In particular, transporting an anyon around one end of the line changes the charge of the anyon according to the action of s.

An example where such topological defects can be realized is in order. Consider a system of qubits forming a square lattice. We introduce a Hamiltonian [11] with commuting four-body interactions, with each plaquette in the lattice contributing a Pauli product term to the Hamiltonian, see Fig. **??**(a):

$$H := -\sum_k A_k, \qquad A_k := X_k Z_{k+\mathbf{i}} X_{k+\mathbf{i}+\mathbf{j}} Z_{k+\mathbf{j}}. \tag{1}$$

Here $k = (i,j)$ indexes the qubit and $\mathbf{i} := (1,0)$, $\mathbf{j} := (0,1)$. The ground subspace is described by the conditions $A_k = 1$. Excitations are localized and gapped: a plaquette k is excited if $A_k = -1$, in which case we say that it holds a quasiparticle. These quasiparticles are abelian anyons [3]. To label quasiparticles with their topological charge, we first have to label plaquettes with two 'colors' as in a chessboard lattice, see Fig. 1(a). Then we can attach a charge e (charge m) to quasiparticles living at dark (light) plaquettes.

Notice that the exchange of e and m labels is trivially a symmetry at the Hamiltonian level, since the choice of dark/light plaquettes is entirely arbitrary. Indeed, this is a symmetry of the corresponding anyon model. In order to implement it in the form of a twist, it is enough to introduce a dislocation in the lattice, as in Fig. **??**(b). Such twists can be created in pairs, braided and fused as if they were anyons, for example by changing the geometry of interactions adiabatically. Indeed, it can be shown [7] that they behave like so called Ising anyons, Ising anyons themselves are not enough for universal quantum computation, they only provide a subset of Clifford gates. Yet, they can be complemented with physically plausible, topologically unprotected, noisy operations to get universal quantum computation [27].

Clifford gates by code deformation. To define a TSCC, the starting point is any three-valent lattice Λ on an oriented closed surface, as in Fig. 2(b). Expanding vertices into triangles and duplicating the existing links, as in Fig. 2(c), produces a new lattice $\bar{\Lambda}$. We place a qubit at each vertex of $\bar{\Lambda}$, and define the so called gauge group $\mathcal{G}$ in terms of a set of 2-local generators, two per qubit. For the marked vertex of Fig. 2(a) these are Z_1Z_2 and Y_2X_3, and this generalizes to other vertices by following the orientation of the lattice. These gauge group generators are all we need to measure in order to recover the error syndrome, but they should not be confused with check operators. These are also local, but related to the plaquettes of the original lattice.

When the original lattice Λ is three-colorable, we recover the standard TSCCs. In these codes error syndromes can

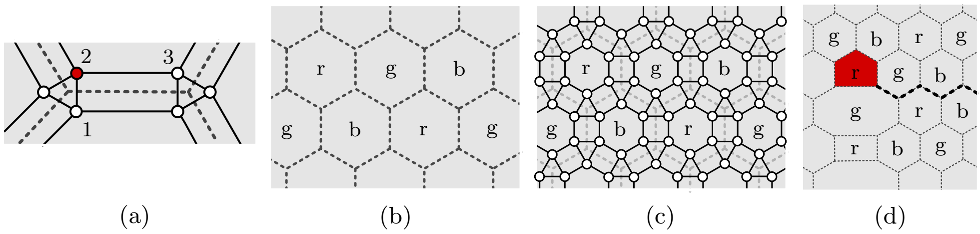

FIG. 2: (a) The three-valent lattice Λ, shown dashed, gives rise to the lattice $\bar{\Lambda}$, in solid lines. (b) A lattice Λ with three colorable faces. (c) The corresponding lattice $\bar{\Lambda}$. (d) A twist, a face with an odd number of links, spoils three-colorability along the dashed line.

be related to an anyon model with three non-trivial topological charges, directly related to the three colors of the lattice. Any permutations of these three charges is a symmetry. When implementing twists, it is enough to consider transpositions where two of the colors are exchanged, and the corresponding geometry for Λ is exemplified in Fig. 2(d).

Such twists turn out to have very powerful consequences. Namely, they allow the implementation of any Clifford gate by suitably braiding the twists [8], which can be done using code deformation.

Conclusions. Twists provide a general tool to improve the computational capabilities of anyonic systems, either from the topological quantum computing perspective or at the level of topological codes. It is particularly interesting the application of twists to TSCCs, which only require 2-local measurements to keep track of errors. Using twists, any Clifford gate can be implemented on these codes by code deformation. Due to the advantages that topological codes exhibit, this makes TSCCs extremely interesting candidates for the implementation of fault-tolerance.

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