On the solution space of Quantum 2-SAT problems

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The Quantum 2-SAT (Q-2-SAT) problem is the natural generalization of the classical 2-SAT problem. It is first studied by S. Bravyi in [1], and shown to be efficiently solvable on a classical computer. In the Quantum 2-SAT problem, a set of projections $\{\Pi_{ij}\}$ is given for $1 \leq i, j \leq n$. Each projection Π_{ij} acts on the i-th and j-th qubit. The question is whether there is a quantum state $|\psi\rangle$ of n qubits such that, for all projections Π_{ij}, $\Pi_{ij}|\psi\rangle = 0$. If such a $|\psi\rangle$ exists, we call it a solution of the Quantum 2-SAT problem; and generally, the solutions constitute a subspace of the n-qubit Hilbert space. Both a single solution and the entire solution space of the Quantum 2-SAT problem are the subject of this investigation.

Before stating our results, let us mention that the Quantum 2-SAT problem is closely related to the so called 2-body frustration-free Hamiltonians in physicists' language. A local Hamiltonian $H = \sum H_j$ is called frustration-free if the ground states of H also minimize the energy of each term H_j. It is easy to see that a Quantum 2-SAT problem $\{\Pi_{ij}\}$ has a solution if and only if the Hamiltonian $H = \sum \Pi_{ij}$ is frustration-free. Moreover, any results obtained in one language can be easily translated in the other.

We prove several results listed below about the solutions of Quantum 2-SAT problem and will discuss them one by one.

S1. *There is always a simple solution which is the product of single- or two-qubit states.*

S2. *The entire solutions space is spanned by tree tensor network states of the same tree structure.*

S3. *Counting the dimension of the solution space is #P-complete.*

Statement S1 tells us the existence of a simple solution. Any state of the product form of single- or two-qubit states is obviously a unique solution of a Quantum 2-SAT problem, whose projections do not even have any overlap. Statement S1 establishes that this kind of trivial solution is unavoidable for a Quantum 2-SAT problem. It generalizes the construction of [1] where the solution found is of a tree tensor network form. The proof of this statement makes uses of the SLOCC classification of three-qubit entanglement [2]. Namely, any three-qubit genuine entanglement is SLOCC equivalent to either GHZ or W state. The SLOCC equivalence plays a role here because two Quantum 2-SAT problems $\{\Pi_{ij}\}$ and $\{L_i^\dagger \otimes L_j^\dagger \Pi_{ij} L_i \otimes L_j\}$ have the same solutions space up to local operation $\bigotimes_j L_j$. The proof is therefore an example where the study of entanglement from quantum information perspective has found an interesting application in many-body physics.

Statement S1 has an important application in measurement-based quantum computing (MBQC) [3]. In MBQC, it is desired to find universal resource state which can occur in natural quantum systems. Favorably, we want a nondegenerate two-body frustration-free Hamiltonian whose unique ground state is a resource for MBQC. This aim has been achieved for spin-$5/2$ [4] and spin-$3/2$ [5] systems. Whereas Statement S1 establishes a negative result for all spin-$1/2$ systems because of the existence of product form solutions which have no power whatsoever of performing universal MBQC.

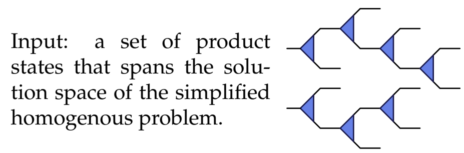
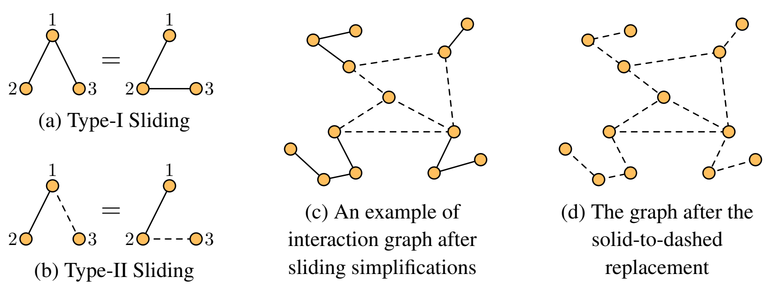


FIG. 1: The general structure of the ground space

Statement S2 characterizes the whole solution space. It extends the result of [1] for a single solution to the whole solution space. The statement opens the door a complete understanding of the Q-2-SAT solution space as in the classical case. The tree tensor networks [6–8] are obtained by applying a series of isometries (from single qubit to two qubits) to a span of products of single-qubit states. See Fig. 1 for an illustration where each triangle represents an isometry. The proof of Statement S2 relies on (1) a case study of the ranks of projections Π_{ij}, where the isometries come from projections of rank 2, and (2) a graphical visualization and modification of the homogeneous case (all projections are rank 1). The solutions space of the homogeneous case is always a span of products of single-qubit states, which constitute the input states in Fig. 1. In the graphical visualization, we use solid and dashed edges to represent entangled and product constraints respectively depending on whether $|\psi\rangle$ in $\Pi_{ij} = |\psi\rangle\langle\psi|$ is entangled or not. Two sliding operations as in Figs. 2a and 2b are used to simplify the graph structure without changing the ground space. This simplification leads to graphs as in Fig. 2c which contains a dashed backbone with several solid tails attached to it.



(a) Type-I Sliding

(b) Type-II Sliding

(c) An example of interaction graph after sliding simplifications

(d) The graph after the solid-to-dashed replacement

FIG. 2: Simplification of the interaction graph

Statement S3 builds up on the previous statement. As counting the dimension of solution space is at least as hard as its classical analog #2-SAT, we know that the problem is #P-hard [9]. However, in the quantum case, even when we know the decision problem of Quantum 2-SAT is in P, it does not easily follow that its counting version is in #P as there may not be any classical way of enumerating all the solutions because of the possible arbitrary entanglement in a solution [10]. Statement S2 alleviates this point, but it is also important that we can enumerate a linearly independent set of solutions in order to count the dimension correctly. This is done by a trick that replaces all the solid tails in Fig. 2c by dashed edges, and reduces the problem to a much simpler case in Fig. 2d where only product constraints are involved.

In contrast to several related works that investigate random instances or generic cases [10–13], all of our results are proved for the general case. The main conclusion is that the solution space of Quantum 2-SAT is of a trivial structure. Both the frustration-freeness and low local dimension is responsible for this phenomena. For frustrated two-body interactions of qubit system, there can be non-trivial entanglement in the ground space (e.g., the Bacon-Shor code [14]), while for frustration-free systems of higher local dimension, the AKLT state [15] has interesting entanglement that can be universal for MBQC [16–18].

In sum, we provide several structural results of the solution space of Quantum 2-SAT problems. In physicists' language, they improve our understanding of quantum spin-$1/2$ systems, and hopefully, will provide ideas for the further research of quantum many-body systems. Related interesting open problems includes the study of the solution space structure for Quantum 2-SAT where each projections concerns a qubit and a qutrit, or even for Quantum 3-SAT. Whether these harder problems have a classically efficient describable solution is at the heart of the complexity of Quantum 2-SAT of qubit-qutrit pair and Quantum 3-SAT [1, 19, 20], and may have implications of whether certain MBQC resource states exist.

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[1] S. Bravyi, arXiv:quant-ph/0602108.
[2] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A, **62**, 062314 (2000).
[3] R. Raussendorf and H. J. Briegel, Phys. Rev. Lett., **86**, 5188 (2001).
[4] X. Chen, B. Zeng, Z.-C. Gu, B. Yoshida, and I. L. Chuang, Phys. Rev. Lett., **102**, 220501 (2009).
[5] J. Cai, A. Miyake, W. Dür, and H. J. Briegel, arXiv:1004.1907.
[6] D. Perez-Garcia, F. Verstraete, M. M. Wolf, and J. I. Cirac, Quant. Inf. Comp., **7**, 401 (2007).
[7] Y.-Y. Shi, L.-M. Duan, and G. Vidal, Phys. Rev. A, **74**, 022320 (2006).
[8] G. Vidal, Phys. Rev. Lett., **91**, 147902 (2003).
[9] L. G. Valiant, Theor. Comp. Sci., **8**, 189 (1979), ISSN 0304–3975.
[10] S. Bravyi, C. Moore, and A. Russell, arXiv:0907.1297.
[11] C. Laumann, R. Moessner, A. Scardicchio, and S. Sondhi, Quant. Inf. and Comp., **10**, 0001 (2010).
[12] C. Laumann, A. Läuchli, R. Moessner, A. Scardicchio, and S. Sondhi, arXiv:0910.2058.
[13] A. Ambainis, J. Kempe, and O. Sattath, arXiv:0911.1696.
[14] D. Bacon, Phys. Rev. A, **73**, 012340 (2006).
[15] I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki, Phys. Rev. Lett., **59**, 799 (1987).
[16] G. K. Brennen and A. Miyake, Phys. Rev. Lett., **101**, 010502 (2008).
[17] T.-C. Wei, I. Affleck, and R. Raussendorf, arXiv:1009.2840 (2010).
[18] A. Miyake, arXiv:1009.3491 (2010).
[19] L. Eldar and O. Regev, in *ICALP* (2008) pp. 881–892.
[20] D. Nagaj and S. Mozes, J. Math. Phys, **48**, 072104 (2007).