The uncertainty principle determines the non-locality of quantum mechanics

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Two central concepts of quantum mechanics are the uncertainty principle, and non-locality. These two fundamental features have thus far been distinct concepts. Here we show that they are inextricably and quantitatively linked. Quantum mechanics cannot be more non-local with measurements which respect the uncertainty principle. In fact, the link between uncertainty and non-locality holds for all physical theories. More specifically, the degree of non-locality of any theory is determined by two factors – the strength of the uncertainty principle, and the strength of a property called "steering", which determines which states can be prepared at one location given a measurement at another. For known inequalities, the steering in quantum mechanics is determined by non-signalling making the degree of non-locality determined by the strength of the uncertainty principle alone. *This submission is based on arXiv:1004.2507, to appear in Science. A newer version is attached which cannot yet appear on the arXiv.*

Non-local correlations in quantum mechanics are much stronger than in a classical world. However, quantum correlations are weaker than what the no-signalling principle demands [1]. It is often asked why quantum mechanics is not more non-local? That is, is there a principle which limits the degree of quantum non-locality? Information-theory [2, 3], communication complexity [4], and local quantum mechanics [5] may provide some rationale why limits on quantum theory may exist. But evidence suggests that many of these attempts provide only partial answers. Here, we take a very different approach and relate the limitations of non-local correlations to two inherent properties of any physical theory: uncertainty relations and our ability to steer.

UNCERTAINTY RELATIONS

The modern approach to quantify uncertainty is to use entropic measures (see full version for details). Such entropic uncertainty relations have the great advantage over tradtional formulations in that they provide bounds that depend only on the measurements, and not on the choice of state, and hence quantify their inherent incompatibility. It has been shown that for two measurements, entropic uncertainty relations do in fact imply Heisenberg's uncertainty relation [6], providing us with a more general way of capturing uncertainty. Such relations also play an important role in quantum cryptography.

Entropic functions are, however, a rather coarse way of measuring the uncertainty of a set of measurements, as they do not distinguish the uncertainty inherent in obtaining any combination of outcomes $x^{(t)}$ for different measurements t. It is thus useful to consider much more fine-grained uncertainty relations consisting of a series of inequalities, one for each combination of possible outcomes, which we write as a string $\vec{x} = (x^{(1)}, \ldots, x^{(n)}) \in \mathcal{B}^{\times n}$ with $n = |\mathcal{T}|$ [16]. That is, for each $\vec{x}$, a set of measurements $\mathcal{T}$, and distribution $\mathcal{D} = \{p(t)\}_t$,

$$P^{\rm cert}(\sigma;\vec{x}) = \sum_{t=1}^{n} p(t)\, p(x^{(t)}|t)_\sigma \leq \zeta_{\vec{x}}(\mathcal{T},\mathcal{D}) \;. \qquad (1)$$

For a fixed set of measurements, the set of inequalities

$$\mathcal{U} = \left\{ \sum_{t=1}^{n} p(t)\, p(x^{(t)}|t)_\sigma \;\leq \zeta_{\vec{x}} \mid \forall \vec{x} \in \mathcal{B}^{\times n} \right\} \;, \qquad (2)$$

thus forms a fine-grained uncertainty relation, as it dictates that one cannot obtain a measurement outcome with certainty for all measurements simultaneously whenever $\zeta_{\vec{x}} < 1$. The values of $\zeta_{\vec{x}}$ thus confine the set of allowed probability distributions, and the measurements have uncertainty if $\zeta_{\vec{x}} < 1$ for all $\vec{x}$. Fine-grained uncertainty relations are directly related to the entropic ones, and have both a physical and an information processing appeal [17]. To characterise the "amount of uncertainty" in a particular physical theory, we are thus interested in the values of

$$\zeta_{\vec{x}} = \max_{\sigma} \sum_{t=1}^{n} p(t) p(x^{(t)}|t)_\sigma \qquad (3)$$

where the maximization is taken over all states allowed on a particular system (for simplicity, we assume it can be attained in the theory considered). We will also refer to the state $\rho_{\vec{x}}$ that attains the maximum as a "maximally certain state". However, we will also be interested in the degree of uncertainty exhibited by measurements on a set of states Σ quantified by $\zeta_{\vec{x}}^{\Sigma}$ defined with the maximisation in (3) taken over $\sigma \in \Sigma$.

Example: Consider the binary spin-$1/2$ observables Z and X. If we can obtain a particular outcome $x^{(Z)}$ given that we measured Z with certainty, i.e. $p(x^{(Z)}|Z) = 1$, then the complementary observable must be completely uncertain i.e. $p(x^{(X)}|X) = 1/2$. If we chose which measurement to make with probability $1/2$ then this notion of uncertainty is captured by the relations valid for all $\vec{x} = (x^{(X)}, x^{(Z)}) \in \{0,1\}^2$,

$$\frac{1}{2} p(x^{(X)}|X) + \frac{1}{2} p(x^{(Z)}|Z) \leq \zeta_{\vec{x}} = \frac{1}{2} + \frac{1}{2\sqrt{2}} \qquad (4)$$

where the maximally certain states are given by the eigenstates of $(X+Z)/\sqrt{2}$ and $(X-Z)/\sqrt{2}$.

NON-LOCAL CORRELATIONS

To state our result, we briefly recall the concept of non-local correlations. Instead of considering measurements on a single system we now consider measurements on two (or more) space-like separated systems traditionally named Alice and Bob. We label Bob's measurements using $t \in \mathcal{T}$, and use $b \in \mathcal{B}$ to label his measurement outcomes. For Alice, we use $s \in \mathcal{S}$ to label her measurements, and $a \in \mathcal{A}$ to label her outcomes [16]. When Alice and Bob perform measurements on a shared state σ_{AB} the outcomes of their measurements can be correlated. Let $p(a,b|s,t)_{\sigma_{AB}}$ be the joint probability that they obtain outcomes a and b for measurements s and t. We can now again ask ourselves, what correlations are possible in nature? In other words, what probability distributions are allowed?

Bell inequalities [7] are used to describe limits on such joint probability distributions. They are most easily explained in their more modern form in terms of a game played between Alice and Bob with questions s,t chosen with probabilities $p(s)$,$p(t)$ and answers a,b. A set of rules determines whether a and b are winning answers. Again for simplicity, we assume the game is unique – for every setting s and outcome a of Alice there is a string $\vec{x}_{s,a} = ({x_{s,a}}^{(1)}, \ldots, {x_{s,a}}^{(n)}) \in \mathcal{B}^{\times n}$ of length $n = |\mathcal{T}|$ that determines the correct answer $b = {x_{s,a}}^{(t)}$ for question t for Bob. We say that s and a determine a "random access coding" [8], meaning that Bob is not trying to learn the full string $\vec{x}_{s,a}$ but only the value of one entry. The case of non-unique games is a straightforward but cumbersome generalisation. To characterize what distributions are allowed, we are generally interested in the winning probability maximized over all possible strategies for Alice and Bob

$$P^{\text{game}}_{\max} = \max_{\mathcal{S},\mathcal{T},\sigma_{AB}} P^{\text{game}}(\mathcal{S},\mathcal{T},\sigma_{AB}) \,, \qquad (5)$$

which is also referred to as a Tsirelson's type bound for the game [9]. The difference between the winning probability $P^{\text{game}}_{\max}$ of a particular theory, and the value that can be achieved classically is thereby referred to as the strength of non-local correlations for this theory. However, the connection we will demonstrate between uncertainty relations and non-locality holds even before this optimization.

Example: The CHSH inequality [10] can be expressed as a game in which Alice and Bob receive binary questions $s,t \in \{0,1\}$ respectively, and similarly their answers $a,b \in \{0,1\}$ are single bits. Alice and Bob win the CHSH game if their answers satisfy $a \oplus b = s \cdot t$. We can label Alice's outcomes using string $\vec{x}_{s,a}$ and Bob's goal is to retrieve the t-th element of this string. For $s = 0$, Bob will always need to give the same answer as Alice in order to win and hence we have $\vec{x}_{0,0} = (0,0)$, and $\vec{x}_{0,1} = (1,1)$. For $s = 1$, Bob needs to give the same answer for $t = 0$, but the opposite answer if $t = 1$. That is, $\vec{x}_{1,0} = (0,1)$, and $\vec{x}_{1,1} = (1,0)$. For the CHSH inequality, we have $P^{\text{game}}_{\max} = 3/4$ classically, $P^{\text{game}}_{\max} = 1/2 + 1/(2\sqrt{2})$ quantum mechanically, and $P^{\text{game}}_{\max} = 1$ for a theory allowing any non-signalling correlations.

STEERING

The third concept, we need in our discussion is steerability, which determines what states Alice can prepare on Bob's system remotely. Imagine Alice and Bob share a state σ_{AB}, and consider the reduced state $\sigma_B = \mathrm{tr}_A(\sigma_{AB})$ on Bob's side.

$$\sigma_B = \sum_a p(a|s)\, \sigma_{s,a} \text{ with } \sigma_{s,a} \in \mathscr{S} \,, \qquad (6)$$

corresponding to an ensemble $\mathcal{E}_s = \{p(a|s), \sigma_{s,a}\}_a$. For all s there exists a measurement on Alice's system that allows her to prepare $\mathcal{E}_s = \{p(a|s), \sigma_{s,a}\}_a$ on Bob's site and for any set of ensembles $\{\mathcal{E}_s\}_s$ that respect the no-signalling constraint, i.e., for which there exists a state σ_B such that (6) holds, we can in fact find a bipartite quantum state σ_{AB} and measurements that allow Alice to steer to such ensembles. We can imagine theories in which our ability to steer is either more, or maybe even less restricted (some amount of signalling is permitted). Our notion of steering thus allows forms of steering not considered in quantum mechanics [11–13] or other restricted classes of theories [14]. Our ability to steer, however, is a property of the set of ensembles we consider, and not a property of one single ensemble.

RESULT

We are now in a position to state our result, details can be found in [17]. For any theory, we find that the uncertainty relation for Bob's measurements (optimal $\mathcal{T}_{\text{opt}}$ or otherwise) acting on the states that Alice can steer to is what determines the strength of non-locality. More specifically,

$$P^{\text{game}}_{\max} = \max_{\{\mathcal{E}_s\}_s} \sum_s p(s) \sum_a p(a|s)\, \zeta^{\sigma_{s,a}}_{\vec{x}}(\mathcal{T}_{\text{opt}}, \mathcal{D}) \,, \qquad (7)$$

and hence the probability that Alice and Bob win the game depends only on the strength of the uncertainty relations with respect to the sets of steerable states. This shows a quantitative relationship between these fundamental concepts which hold for any theory. In quantum theory, can we remove the dependence on steering? Alice needs to be able to prepare the ensemble $\{p(a|s), \rho_{\vec{x}_{s,a}}\}_a$ of maximally certain states on Bob's system. In general, it is not clear that the maximally certain states for the measurements which are optimal for Bob in the game can be steered to. But this can be achieved in cases where we know the optimal strategy. For all XOR games, that is correlation inequalities for two outcome observables (which include CHSH as a special case), as well as other games where the optimal measurements are known we find that the states which are maximally certain can be steered to [17]. We thus have that the uncertainty relations for Bob's optimal measurements give a tight bound

$$P^{\text{game}}_{\max} = \sum_s p(s) \sum_a p(a|s) \zeta_{\vec{x}}(\mathcal{T}_{\text{opt}}, \mathcal{D}) \,, \qquad (8)$$

where we recall that $\zeta_{\vec{x}}$ is the bound given by the maximization over the full set of allowed states on Bob's system. It is an open question whether this holds for all games in quantum mechanics.

An important consequence of this is that any theory that allows Alice and Bob to win with a probability exceeding $P^{\rm game}_{\rm max}$ requires measurements which do not respect the fine-grained uncertainty relations given by $\zeta_{\vec{x}}$ for the measurements used by Bob (the same argument can be made for Alice). Even more, it can lead to a violation of the corresponding min-entropic uncertainty relations [17]. For example, if quantum mechanics were to violate CHSH more it would need to do so with measurements which are more certain. The same measurements would need to violate the min-entropic uncertainty relations [15]. This relation holds even if Alice and Bob were to perform all-together different measurements when winning the game with a probability exceeding $P^{\rm game}_{\rm max}$: for these new measurements there exist analogous uncertainty relations on the set Σ of steerable states, and a higher winning probability thus always leads to a violation of such an uncertainty relation. Conversely, if a theory allows any states violating even one of these fine-grained uncertainty relations for Bob's (or Alice's) optimal measurements on the sets of steerable states, then Alice and Bob are able to violate the Tsirelson's type bound for the game.

CHSH EXAMPLE

Although the connection between non-locality and uncertainty is more general, we examine the example of the CHSH inequality to gain some intuition on how uncertainty relations of various theories determine the extent to which the theory can violate Tsirelson's bound [17]. Briefly, in quantum theory, Bob's optimal measurements are X and Z which have uncertainty relations given by $\zeta_{\vec{x}_{s,a}} = 1/2 + 1/(2\sqrt{2})$ of (4). Thus, if Alice could steer to the maximally certain states for these measurements, they would be able to achieve a winning probability given by $P^{\rm game}_{\rm max} = \zeta_{\vec{x}_{s,a}}$ i.e. the degree of non-locality would be determined by the uncertainty relation. This is indeed the case – if Alice and Bob share the singlet state then Alice can steer to the maximally certain states by measuring in the basis given by the eigenstates of $(X+Z)/\sqrt{2}$ or of $(X-Z)/\sqrt{2}$. For quantum mechanics, our ability to steer is only limited by the no-signalling principle, but we encounter strong uncertainty relations limiting non-locality.

On the other hand, for classical deterministic mechanics we have no uncertainty relations on the full set of deterministic states, but our abilities to steer to them are severely limited, giving $P^{\rm game}_{\rm max} = 3/4$. This bound also holds for (randomized) local hidden variable theories. Finally, there are theories which are maximally non-local, yet remain no-signalling [1]. These have no uncertainty, i.e. $\zeta_{\vec{x}_{s,a}} = 1$, but unlike in the classical world we still have perfect steering, so they win the CHSH game with probability 1.

For any physical theory we can thus consider the strength of non-local correlations to be a tradeoff between two aspects: steerability and uncertainty. In turn, the strength of non-locality can determine the strength of uncertainty in measurements. However, it does not determine the strength of complementarity of measurements [17]. The concepts of uncertainty and complementarity are usually linked, but we find that one can have theories which are just as non-local and uncertain as quantum mechanics, but which have less complementarity. This suggests a rich structure relating these quantities, which may be elucidated by further research in the direction suggested here.

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[16] Without loss of generality we assume that each measurement has the same set of possible outcomes, since we may simply add additional outcomes which never occur.
[17] Full version attached.